CHAPTER 10

Notes on the Origin of the First Definition of Zero Consistent with Basic Physical Laws

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Abstract

If mathematics is the language the universe was written in and mathematics is discovered rather than invented, then Brahmagupta's 628 CE Sanskrit text first defines a zero as old as the universe. Brahmagupta's definition of zero as a sum of equal and opposite negative and positive quantities is the first scientific definition of zero found to be consistent with laws of motion and particle physics. However, having originated in the East, the full power of India's symmetric and scientific zero failed to migrate West via the medieval Arabic world to Renaissance Europe. Thus, only a trivial mathematical concept of zero emerged, representing either nothing on its own or an arithmetical placeholder used alongside other numbers. Zero as an arbitrary midpoint for measurement purposes within a single quantity is similarly trivial, whether it be centuries BCE or CE, or a measure above or below a surveyor's foundation line as used for construction in ancient Egypt. Thus, the oldest original extant definition of zero compatible with laws of physics describing our universe originated with Brahmagupta in India 628 CE. Yet long lost in transmission and translation, the sad aftermath in classrooms today is a zero in which negatives to its left are arbitrarily less than positives to its right.

Keywords

Brahmagupta – India – mathematics – zero – negatives – positives – physics – *śūnya*

1 Introduction

The entry for 'zero, n. and adj.' in the online *Oxford English Dictionary* (OED) defines zero in the sub-section 'Mathematics' as 'The absence of quantity considered as a number; nought'. This is followed by, 'The earliest example of zero considered as a number in its own right occurs in a manuscript by

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Indian mathematician Brahmagupta (598–668) dated to the seventh cent.' So, it seems we have our first mathematical definition of zero as a number, written in Sanskrit text as *śūnya* (pronounced shoonya) in 628 CE. For the first zero symbol as a circle, a popular meme on the internet is that Bhāskara I (*c.*600–*c.*680) the next year in 629 CE, in *Aryabhatiyabhashya*, a commentary on Aryabhata's work, was the first to use a circle for India's zero. Supposedly, Bhāskara I wrote 'nyâsaśca sthânânâm 0000000000', meaning 'writing down the places we have 0000000000' (Datta and Singh, 1962). Alas, such a simple finding for the first mathematical use of the symbol 0 is merely a mirage. More profound for mathematics students today, Brahmagupta's seventh century mathematical definition of zero (*śūnya*), led to nothing. The original mathematical zero Brahmagupta defined, as old as the universe, remains shrouded in equal parts philosophical sense and mathematical nonsense, while its symmetry continues to unlock secrets of our universe.

Just as the OED provides a trivial definition of zero, elsewhere we find definitions of zero falsely attributed to Brahmagupta such as 'the result of subtracting any number from itself' (Barrow, 2001, p. 38). This 'nothing remaining as a result of subtraction' and placeholder notion may have been an idea that reached the Arabic world on its way to Europe. Yet, as will be noted, Brahmagupta's zero is consistent with ideas such as conservation of matter and energy and Newton's third law, for every action there is an equal and opposite reaction.

On zero we read, 'We know that *śūnya* traveled from India to Europe via the *algorismus* texts, starting tenth century CE, and that the epistemological assimilation *śūnya* required some five to six hundred years' (Raju, 2007, p. 95). Yet, India's original mathematical definition of zero and its elementary applications born from empirical physical foundations were never fully transmitted from the ancient East to Western classrooms today.

Only the 'nothing' and placeholder concepts of zero made their way from India to the Arabic world and from there into Western pedagogies. For nearly all this time, zero was not considered a number mediating equal and opposite quantities in either the Arabic world or Europe. Around 300 BCE Euclid defined a number as 'a multitude composed of units' (Heath & Euclid, 1908, p. 277). So, zero was not alone in its struggle to be considered a number. For most of our Western history of mathematics via the Greeks, one was also not formally considered a number. In number theory, zero was only granted formal number status in the twentieth century. Yet accepting India's zero as both placeholder and a number in its own right still leaves zero's most important and powerful attribute, symmetry, in the dark.

In what might come to be described as a 'Black Swan' event (Taleb, 2007), Brahmagupta's original definition of zero failed to find its way out of India in time for the arrival of the printing press in the West. England, in particular, exported mathematics books to its settlements and colonies, with explanations largely built upon Greek (Euclidean) foundations. These foundations did not feature one as a number and both zero and negatives were absent. Thus, neither Brahmagupta's *zero* nor *one* came to be included in modern algorithmic definitions of multiplication and exponentiation (Crabtree, 2017a, p. 14). The aftermath has been a pedagogical Dark Age that continues to this day. Extraordinary claims require extraordinary evidence. Thus, we shine new light upon Brahmagupta's definition of zero, to see its embedded twin shadows of opposing *negative* and *positive* quantities emerge. Having a pedagogical problem with zero in 1968, the author literally began rebuilding elementary mathematics from zero in 1983. To the reader, the author's call for a major rewrite of elementary mathematics curricula might come as a shock. Yet, in hindsight, it might one day be seen as having been inevitable. Google reports hundreds of matches for the exact phrases, 'crisis in mathematics education' and 'fear of mathematics'. The social proof is real. When it comes to Brahmagupta's original mathematical definition of zero, in which negative is equal and opposite to positive (rather than less than positive), has history taught us nothing?

2 On Mathematics and Progress

Interviewed on the documentary *Infinite Secrets* about a manuscript of Archimedes that went missing, Dr Chris Rorres, Professor Emeritus of Mathematics at Drexel University, said: 'If we had been aware of the discoveries of Archimedes hundreds of years ago, we could have been on Mars today, we could have developed the computer that is as smart as a human being today, we could have accomplished all of the things that now people are predicting for a century from now' (Rorres, 2003).

Similarly, had Brahmagupta's definition of zero been understood and documented in the Arabic world within the writings of al-Khwārizmī and others, algebraic innovations might have occurred centuries sooner. Whether or not the absence of Brahmagupta's definition of zero in both the Arabic world and Europe held back scientific progress a century is just speculation. However, having deciphered Brahmagupta's cipher, many difficulties of great mathematicians from Diophantus to Descartes could have been avoided.

3 On the Pedagogical Absence of Zero and One

3.1 *Zero's Absence*

At school in 1968, age seven, the author knew how to count, add, and subtract. The teacher of Grade 2C, Miss Collins, said multiplication was like repeated addition. So, her explanation should have been simple, yet it wasn't. After writing 2×3 on the blackboard, Miss Collins asked her class, 'What is two added to itself three times?' Because I could understand that 2 added to 1 three times is seven ($1 + 2 + 2 + 2$), when Miss Collins chose me to answer 'What is two added to itself three times?' my answer was eight ($2 + 2 + 2 + 2$). Surprisingly, the English language rhetorical explanation given for 2×3 followed verbatim has led to eight for centuries. Miss Collins drew three 'hops' of 2 on the blackboard number line that landed on 6. However, the three hops started at India's zero. Pedagogically, explaining 2×3 via repeated addition requires more precise statements such as 'three twos added together' or 'two added to zero (not itself) three times', which is $0 + 2 + 2 + 2$.

Many years later, it became obvious that Miss Collins' explanation of multiplication was a paraphrase of an incorrect translation of Euclid's 300 BCE definition of multiplication. This reads, 'a number is said to multiply a number when that which is multiplied is added to itself as many times as there are units in the other, and thus some number is produced' (Heath & Euclid, 1908, p. 278). Today, Euclid's definition of multiplication has been modernized (yet remains incorrect) to read 'to multiply **a** by integral **b** is to add **a** to itself **b** times' (Borowski & Borwein, 2012, p. 376). However, India's zero is needed in the definition of multiplication. Without a prior knowledge of multiplication facts, the English definition of multiplication dating back to 1570 would have us believe 2×1 does not equal 1×2. The former would be 'two added to itself once' which is four ($2 + 2$), while the latter would be 'one added to itself twice' which is three ($1 + 1 + 1$). Yet the Commutative Law ($\boldsymbol{ab} = \boldsymbol{ba}$) means the products of 2×1 and 1×2 must be the same. Returning India's zero to its rightful place in the English definition of multiplication fixes the anomaly, as 2×1 and 1×2 become two added to zero once and one added to zero twice respectively, both of which equal two. India's zero needs to be included in the integral definition of $\boldsymbol{ab}$. Thus, $\boldsymbol{a}$ multiplied by $\boldsymbol{b}$ is *either* $\boldsymbol{a}$ added to ZERO $\boldsymbol{b}$ times, *or*, $\boldsymbol{a}$ subtracted from ZERO $\boldsymbol{b}$ times, according to the sign of $\boldsymbol{b}$.

As we will see, Brahmagupta's scientific symmetric zero definition failed to be carried across cultures. We might think elementary mathematics has been carefully assembled and improved over time, yet the pedagogies in place today have emerged in large part due to ignorance of symmetric ideas in the

East. The writings of Bhāskara II (1114–1185) and Brahmagupta (598–668) on zero, negatives, and positives were translated into English language books too late to have any impact on pedagogies (Strachey, 1813; Colebrook, 1817; Taylor, 1816).

An author and former mathematics teacher asked:

> Do you really think that children were supposed to learn a problem like 1–2 five years after learning 2–1? The unwarranted separation of positive and negative numbers is the most glaring symptom that we merely mapped our historical misunderstanding of zero and negative numbers onto our math education, then called it a day. (Singh, 2021)

Englishmen published primary-level mathematics pedagogies built largely upon Greek foundations 1,000 years older than the writings of Brahmagupta. The British Empire then exported these out-of-date explanations of mathematics to their settlements and colonies. So, as English became the world's de facto language, the sub-optimal pedagogies sold in sixteenth/seventeenth-century England came to be disliked by customers (children) worldwide today. French pedagogies were superior, as you might expect from a country that would later champion the base-10 metric system. By way of example, around 200 years ago, an American mathematics professor wrote: 'The first principles, as well as the more difficult parts of Mathematics, have, it is thought, been more fully and clearly explained by the French elementary writers, than by the English' (Farrar, 1818, Preface).

4 Brahmagupta's Zero-Sum Definition

Chapter 18 of Brahmagupta's *Brāhmasphuṭasiddhānta*, was about algebra (*Kuttaka*). The section containing Brahmagupta's original definition of *śunyā* was titled *Dhanarṇa Śunyānām Samkalanam*, or 'calculations dealing with quantities bearing positive and negative signs and zero' (Prakash, 1968, p. 200). This section detailed the laws of sign for positives, negatives and zero. Within his laws of addition (*saṅkalana*), Brahmagupta defined zero as the sum of equal positive and negative (Plofker, 2009, p. 151). As an astronomer, symmetry was central to Brahmagupta's calculations. If for example, North was positive then South was negative. Zero's role as an additive identity was emphasized to the extent that even zero plus zero is zero was noted in Brahmagupta's Sanskrit Laws of addition which contain his definition of zero (Figure 10.1).

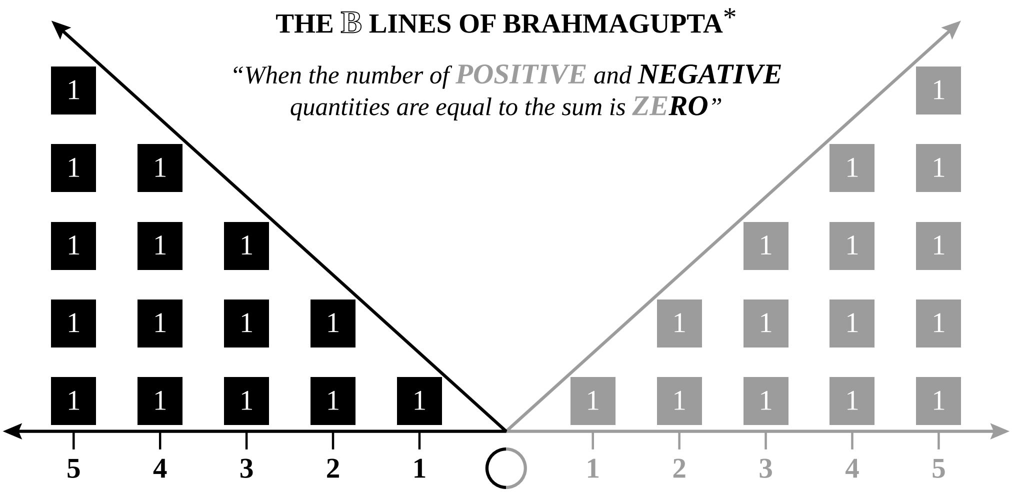
Brahmagupta's 5 Addition Sutras

धनयोर् धनम् ऋणमृणयोः धनर् णयोरन् तरं समैक् यं खम्
ऋणमैक् यं च धनमृणधनशून् ययोः शून् ययोः शून् यम्

AS1	positive plus positive is positive
AS2	negative plus negative is negative
AS3	positive plus negative is the difference between the positive and negative
AS4	when positive and negative are equal the sum is zero
AS5	positive plus zero is positive negative plus zero is negative zero plus zero is zero

FIGURE 10.1 Brahmagupta's five addition sutras

Thus, we can anachronistically depict the central theme of India's zero mediating equal and opposite line magnitudes in the diagram below (Figure 10.2).



*598–668 CE INDIA

FIGURE 10.2 India's zero represented the least magnitude among equal yet opposing magnitudes

In 1039 CE, Śrīpati's treatise *Siddhāntaśekhara* also defined zero as the sum of two equal negative and positive numbers (Datta & Singh, 1962, p. 21). Thus, we read 'sunya (zero) is neither positive nor negative but forms the boundary line between the two kinds, being the sum of two equal but opposite quantities' (Joseph, 2016, p. 108). It is this zero, as old as the universe where forces exist as equal and opposite pairs, that waits to be fully incorporated into our concept of zero in mathematics classrooms today. The integer inequality symbols $<$ and $>$ (Oughtred, 1631) predated the first appearance of a number line (Wallis, 1685) with numbers either side of zero. Sadly, we are thus taught two negatives are greater than five negatives and told to write ${}^{-}2 > {}^{-}5$ which is contrary to the laws of physics.

China's approach was similar, yet with columns on a counting board being used to separate numbers into places; a zero symbol was not required as a placeholder. When either a positive number or negative number was to be subtracted from an empty place in Chinese arithmetic (Martzloff, 2006), the rules were as follows:

> [If a] positive [rod] does not have a vis-&-vis (i.e., a number facing it) it is made negative (a positive number subtracted from nothing becomes negative). Conceptually, this is $[\ ] - [{}^{+}3] = [{}^{-}3]$. Then, the rules continue, [If a] negative [rod] does not have a vis-&-vis, it is made positive (i.e., a negative number subtracted from nothing becomes positive). Conceptually, this is $[\ ] - [{}^{-}3] = [{}^{+}3]$.

In the second century BCE, China's negative and positive numbers were depicted with black and red rods being equal yet opposite in value (Shen, 1999). Therefore, a similar anachronistic diagram may emerge that again depicts a Chinese absence of number in a column with a conceptual midpoint between negative and positive as shown in Figure 10.3.

Following on from rod numeral arithmetic, the subsequent use of the abacus meant the 0 symbol for zero was delayed by centuries in China, first appearing in *Mathematical Treatise in Nine Sections* by Qín Jiǔsháo in 1247 CE.

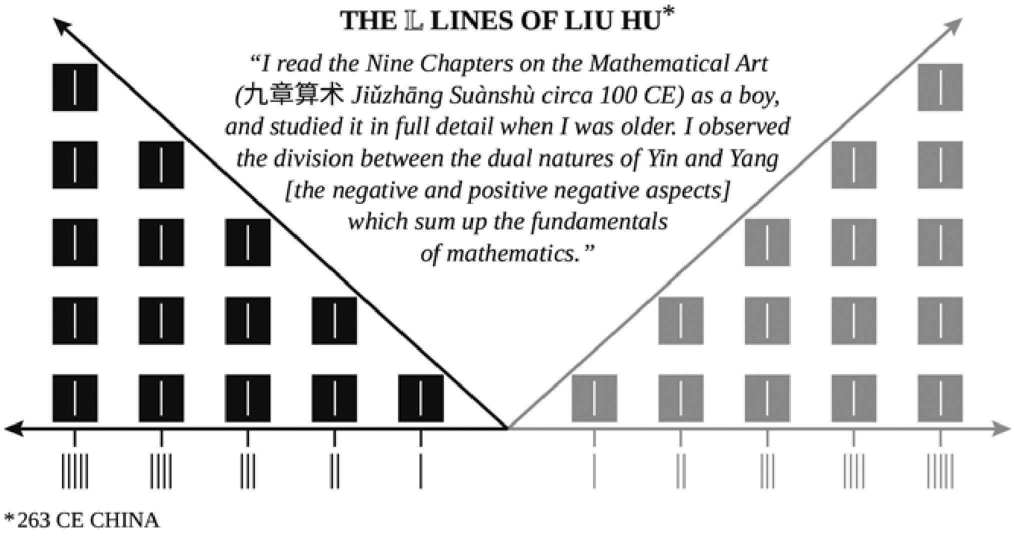


FIGURE 10.3 China's equal and opposite rod numeral system had empty places rather than a zero

5 Brahmagupta's Laws of Zero

Just as *left* and *right* cannot exist without a *center*, you cannot have the relational concept of *positive* and *negative* without a *center*. Similarly, with *above* and *below*, zero is neither term but formed from both. Brahmagupta's zero exists between positive and negative because he defined it as the sum of both (equal) positive and negative. It is this aspect, being a mathematical sum, that not only qualifies zero as a number but gives it the capacity to be the sum of all numbers in the set of real numbers. Zero has the capacity to be both the void and the infinite. As an astronomer first and mathematician second, Brahmagupta dealt with relationships between quantities. Numbers, being isomorphic to quantities, were merely tools of astronomers, as were their instruments. Zero, to the Indian scientist, acted as a reference point from which counts and measurements of quantities were made. Brahmagupta's 18 sūtras of symmetry (excluding division by zero), are shown in Table 10.1 (Dvivedin, 1902, p. 309; Plofker, 2009, p. 151).

TABLE 10.1 Brahmagupta's 18 sūtras of symmetry for zero, positive and negative

Addition 1. Positive plus positive is positive. 2. Negative plus negative is negative. 3. Positive plus negative is the difference between the positive and negative. 4. When positive and negative are equal the sum is zero. 5. Positive plus zero, is positive, negative plus zero is negative and zero plus zero is zero.	*Subtraction* 6. A smaller positive subtracted from a larger positive is positive. 7. A smaller negative subtracted from a larger negative is negative. 8. If a larger negative or positive is to be subtracted from a smaller negative or positive, the sign of their difference is reversed, negative becomes positive and positive becomes negative. 9. A negative minus zero is negative, a positive minus zero is positive and zero minus zero is zero. 10. When a positive is to be subtracted from a negative or a negative from a positive, then it is to be added.
Multiplication 11. The product of a negative and a positive is negative. 12. The product of negative and negative is positive. 13. The product of two positives is positive. 14. The product of zero and a negative, of zero and a positive, or of two zeros is zero.	*Division* 15. A positive divided by a positive is positive. 16. A negative divided by a negative is positive. 17. A positive divided by a negative is negative. 18. A negative divided by a positive is negative.

6 On Brahmagupta's Zero and the Foundations of Physics

Within Brahmagupta's quantitative laws, positive and negative co-exist as equal terms. His laws were likely derived from empirical observation and the need to solve problems, an approach similar to the 'scientific method'. Thus, as an astronomer Brahmagupta's ancient treatment of zero, negatives, and positives is consistent today with the laws of physics. Just as India's mathematics treated negative and positive as equal and opposite, today we accept Newton's third law – for every action there is an equal and opposite reaction. Thus, the zero-sum game of mathematics leads to innovations in physics. An example is Paul Dirac's predicted discovery of the positron as the antiparticle for the electron. When one positron and one electron meet, they cancel each other out, effectively summing to zero. Brahmagupta's zero-sum logic can also be associated with the physical law of conservation of matter and energy. The laws of physics and the laws of mathematics that describe quantitative relationships require a correct definition and understanding of zero. Numbers and quantities are symmetric around zero, yet the history of mathematics has essentially been asymmetric, being built upon half the system, the positive. The realization half of Brahmagupta's algebraic laws of sign (involving negative quantities) were mostly missed in the Arabic world appears to have rarely dawned on Western historians and teachers today who bear the brunt of the incomplete transmission of India's zero to the West.

7 The Placeholder Zero in Brahmagupta's Arithmetic

In Brahmagupta's Chapter 12 on arithmetic (*Ganita*), we read, 'He, who distinctly and severally knows addition and the rest of the 20 logistics, and the eight determinations including measurement by shadow, is a mathematician' (Colebrooke, 1817, p. 277). The content is far broader than most arithmetic curriculums today, as shown in Table 10.2 (Pingree, 1981, p. 57).

TABLE 10.2 Brahmagupta's 20 Arithmetical operations and 8 determinations

parikrama 20 fundamental operations	saṅkalita addition	vyavakalita subtraction	pratyutpanna multiplication	bhāgahāra division
varga square	vargamūla square root	ghana cube	ghanamūla cube root	bhāṇḍapratibhāṇḍa barter
pañcajātayaḥ operating on fractions (5 rules)	trairāśika rule of three	vyastatrairāśika inverse rule of 3	and … rules of 5, 7, 9, and 11 terms (4 rules)	vyavahāra 8 determinations mixtures, series, plane geometry, solid geometry, stacks, sawn lumber, mounds of grain and shadow problems.

It is known Brahmagupta's writings on arithmetic (or a derivative of it) made its way to the Arabic world. Zero made its way too, yet as a placeholder, not a number, and not as a *sum* of equal positive and negative. The Arabs embraced the Hindu system of base-10 numeration in which nine distinct letters (numerals) 9, 8, 7, 6, 5, 4, 3, 2, and 1 were recycled to represent different quantities in different places by virtue of zero. In an additive base-10 numeration system, III represents three, while in a multiplicative base-10 system, 111 means one hundred and eleven.

The most obvious advantage of India's zero, which enabled the base-10 multiplicative system over an additive system, such as the numerals of the Roman Empire, is compactness. For example, the number 888 in the Indian system is DCCCLXXXVIII in the Roman system, where zero as a symbol does not exist. While we read from left to right, Arabic is read from right to left. The number 6 does not change with direction, yet VI read left-right could be mistaken for 4 if written IV in a right-left context. Similar confusion may have existed with 11 written XI, being confused with IX which is 9. The single digit base-10 Hindu numeration system was the innovation the Arabs embraced – not India's number zero. As will be revealed, zero was treated as a vital placeholder enabler of the system, yet was not considered a number.

How did Brahmagupta's ideas on zero, positives and negatives reach the West? The story of mathematics has India's zero as documented by Brahmagupta being embraced by the Arabic world. Traders in Northern Africa then passed India's concept of zero onto Leonardo Pisano (Fibonacci), who helped introduce both Hindu and Arabic mathematics to Europe at the start of the thirteenth century. Alas, this story does not agree with the evidence.

Given Brahmagupta's Hindu arithmetic noted in Table 10.2 did *not* discuss the rules of operating with positives, negatives, and zero, such information appears to have not been fully appreciated in the Arabic world. As will also be discussed, it appears Hindu algebra, which featured both positives and negatives, may have been transmitted orally to the Arabic world yet al-Khwārizmī and others seemingly failed to either document it or benefit from it.

From the writings of Brahmagupta, we will explore the partial transmission of his symmetric mediating zero via several important Arabic influencers, (al-Khwārizmī, al-Uqlīdisī and Kūshyār ibn Labbān). Then we explore the influential writings of the Italian Leonardo Pisano, whose 1202 CE book *Liber Abaci* appears to have helped bring about the demise of Gerbert's abacus, which we discuss shortly. Zero the placeholder made its way West (Figure 10.4), while zero as defined and applied as the sum of equal positive and negative remained in the East.

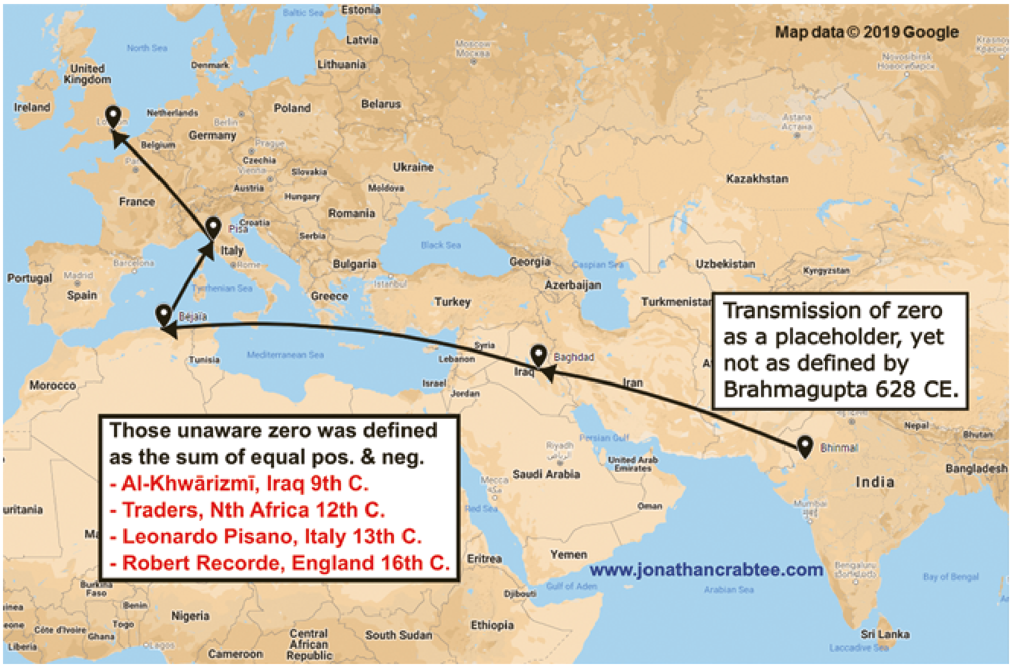


FIGURE 10.4 The transmission of zero as a placeholder from East to West

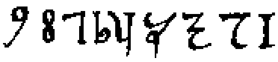


FIGURE 10.5
The oldest extant Arabic numerals to be used in Europe (without zero)

It should be noted that the above timeline depicting the westward travels of zero as a placeholder was the second introduction of (Hindu) Arabic numerals into Europe. The oldest extant Arabic numerals in Europe (without zero) are found in the *Codex Vigilanus* of 976 CE, as seen in the Wikimedia Commons image (Figure 10.5). These came via Muslims in Spain, who came in contact with the Frenchman Gerbert of Aurillac (later Pope Sylvester II) who promoted the use of Arabic numerals on European counting boards.

From the late tenth to the early thirteenth century base-10 positional calculation was performed on a counting board or abacus with Arabic numerals (Ifrah, 2000, p. 580). Given the counting boards had fixed columns, within which counters (called apices) labeled with numerals would be placed, zero as a placeholder was not required. Thus, for more than 200 years 1, 2, 3, 4, 5, 6, 7, 8, and 9 were used in Europe without zero being needed as a placeholder, as revealed by the Wikimedia Commons image Apices of the modern age (Figure 10.6).

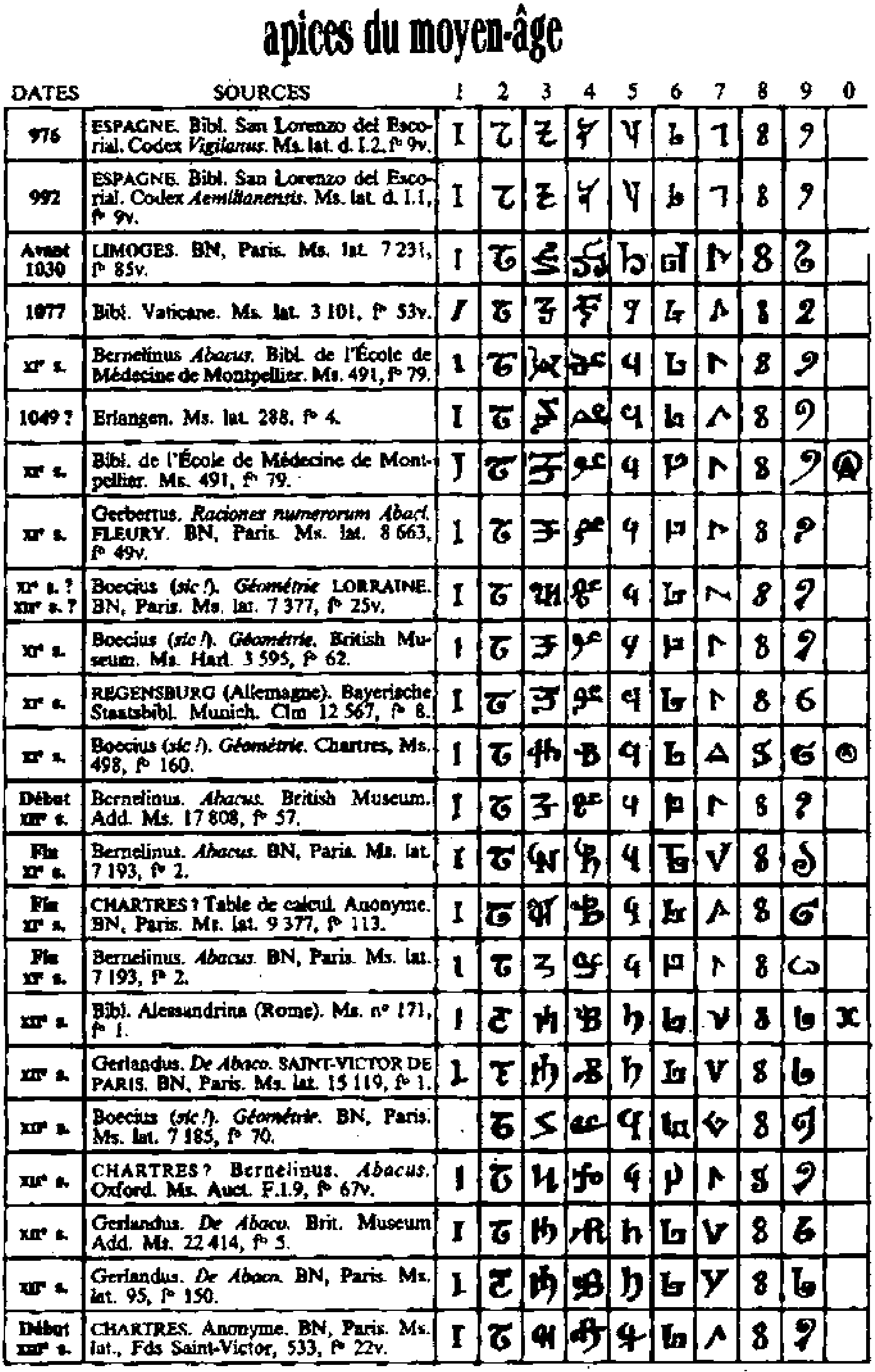


FIGURE 10.6 The various symbols used on apices (counters) for base-10 calculations

8 On the Absence of Brahmagupta's Zero in the Arabic World

8.1 *On al-Khwārizmī's Arithmetical Absence of Brahmagupta's Zero*

Popular writers on the history of mathematics would have readers believe the ideas of Brahmagupta, such as the definition of zero and the 'laws of sign' for the four arithmetical operations, were mastered by al-Khwārizmī (780–850 CE). The legend perpetuated by such writers is that al-Khwārizmī subsequently wrote a book on India's arithmetic around 820 CE, known via its Latin translation as *Algoritmi de numero Indorum*.

Notably, the translator's introduction to al-Khwārizmī's text makes no mention of zero in his introduction. It also states one is not a number. Early in a Latin translation, we read (Crossley and Henry, 1990, pp. 110–111):

> Algorizmi said: since I had seen that the Indians had set up IX symbols in their universal system of numbering, on account of the arrangement which they established, I wished to reveal, concerning the work that is done by means of them, something which might be easier for learners if God so willed. If, moreover, the Indians had this desire and their intention with these IX symbols was the reason which was apparent to me, God directed me to this. If, on the other hand, for some reason other than that which I have expounded, they did this by means of this which I have expounded, the same reason will most certainly and without any doubt be able to be found. And this will easily be clear to those who examine and learn.
>
> So they made IX symbols, whose are these: (9 8 7 6 5 4 3 2 1). There is also a variation among men in regard to their forms: this variation occurs in the form of the fifth symbol and the sixth, as well as the seventh and the eighth. But there is no impediment here. For these are marks indicating a number and the following are the forms in which there is that variation: (5 4 3 2). And already I have revealed in the book of algebra and almuqabalah, that every number is composite and that every number is put together above one. Therefore one is found in every number and this is what is said in another book of arithmetic. Because one is the root of all number and is outside number. It is the root of number because every number is found by it. But it is outside number because it is found by itself, i.e., without any other number.

Given al-Khwārizmī's translator appears so insistent 'one was outside number', it is understandable why zero did not rate a mention in his introduction. Al-Khwārizmī's legacy, whether intended or not, is zero, which acted as a placeholder, was not a number. While al-Khwārizmī covers the four basic operations of addition, subtraction, multiplication, and division, negative numbers are absent from his book on Hindu arithmetic. Yet, in Brahmagupta's laws of the four basic arithmetical operations (noted in his algebra, not his arithmetic), negatives are equally as prevalent as positives.

Comparing al-Khwārizmī's approach to Brahmagupta's, we read (Rashed, 2009, p. 77):

> Once again al-Khwārizmī differs from Brahmagupta, this time in not employing any abbreviation. Also he avoids using 'negative' numbers or simply a [larger] number subtracted from a smaller one, or from zero, whereas Brahmagupta, like other Indian mathematicians before him, does not hesitate to make use of such [negative] numbers. It is difficult to imagine that al-Khwārizmī, if he had read this chapter (i.e., chapter 18 of Brahmagupta's *Brāhma Sphuta-siddhānta*) would not have been able to profit by it, even if only to shorten the presentation of his work.

8.2 *On al-Khwārizmī's Algebraic Absence of Brahmagupta's Zero*

Several years later, al-Khwārizmī was said to have written his landmark book, الكتاب المختصر في حساب الجبر والمقابلة, *al-Kitāb al-mukhtasar fī hisāb al-jabr wa'l-muqābala* (The Compendious Book on Calculation by Restoration and Confrontation). (Sometimes the phrase *Completion and Balancing* is used instead of *Restoration and -Confrontation*.) The word *al-jabr* in the title gave us the word algebra while algorithm (via algorism) is derived from the name al-Khwārizmī. Writing around 200 years after Brahmagupta, it is often assumed al-Khwārizmī wrote first on arithmetic, then on algebra. However, this is not the case. Al-Khwārizmī wrote his book on algebra around 820 CE and followed this up with his book on Hindu arithmetic around 825 CE. Al-Khwārizmī refers readers to his book on algebra in his arithmetic book. Notably, al-Khwārizmī's algebra has little connection to the earlier mathematics of India.

We find possible knowledge of Hindu astronomy and definite knowledge of Hindu arithmetic in the writings of al-Khwārizmī, yet not Brahmagupta's algebraic laws, which featured negative quantities as much as positive quantities. We read (Rashed, 2009, p. 79):

> Whether we are concerned with concepts or procedures, the many divergences indicate that, even if al-Khwārizmī did know books by Āryabhaṭa and Brahmagupta, he had read them only for astronomy and, perhaps, for arithmetic. In any case, reading them had no effect on his conception of

> algebra and it exercised no influence on the techniques he employed in the discipline. The style of the mathematical reasoning that is at work in al-Khwārizmī's algebra has nothing to do with what we encounter in the work of his (Indian) predecessors.

Accepting negative numbers and the identity elements (zero and one) as numbers is critical to the development and evolution of mathematics. However, it is evident from the translations of al-Khwārizmī's text, he used zero as a placeholder yet not as a number. The Arabic world did not see one as a number (being the unit of count or measure) and did not document India's laws of sign for positive and negative numbers. (Like Diophantus of Alexandria centuries before, al-Khwārizmī was aware a positive number with a subtraction multiplied by a positive number also with a subtraction results in a positive number being added as an adjustment.)

8.3 *Balancing Algebraic Equations without India's Zero*

The words *al-jabr wa'l-muqābala* (literally restoration and confrontation) in al-Khwārizmī's book on algebra have been loosely translated as 'balancing an equation' (Devlin, 2012, p. 25). In an environment in which weights and scales often determined the cost of goods at marketplaces, it may have been that al-Khwārizmī had this metaphor in mind for his algebraic equations. If so, then what scale could balance against zero? Without symbols presented here for clarity, al-Khwārizmī provided solutions for linear and quadratic equations involving combinations of $\boldsymbol{ax^2}$ (squares) and $\boldsymbol{bx}$ (roots) and $\boldsymbol{c}$ (numbers) involving positive rationals. If subtraction was involved, it would be eliminated by adding the subtracted term onto both sides to keep the equations balanced. For example, $\boldsymbol{ax^2 = bx - c}$ became $\boldsymbol{ax^2 + c = bx}$. Al-Khwārizmī's six standard types of balanced equations are depicted below in Figure 10.7.

From al-Khwārizmī's six types of balanced equation, his three normal forms were:

1) $\boldsymbol{x^2 + bx = c}$ square + roots = number (*where roots are the side of a square*)
2) $\boldsymbol{x^2 = bx + c}$ square = roots + number, and
3) $\boldsymbol{x^2 + c = bx}$ square + number = roots

To popularize a little-known branch of mathematics, algebra, separate to both geometry and arithmetic is a remarkable achievement. Yet if you don't treat zero as a number, which combinations of positive and negative terms can equal, you won't solve equations of the form $\boldsymbol{ax^2 + bx + c = 0}$. Al-Khwārizmī's modus operandi was to eliminate subtracted terms and confront only positive terms on opposing sides to balance equations.

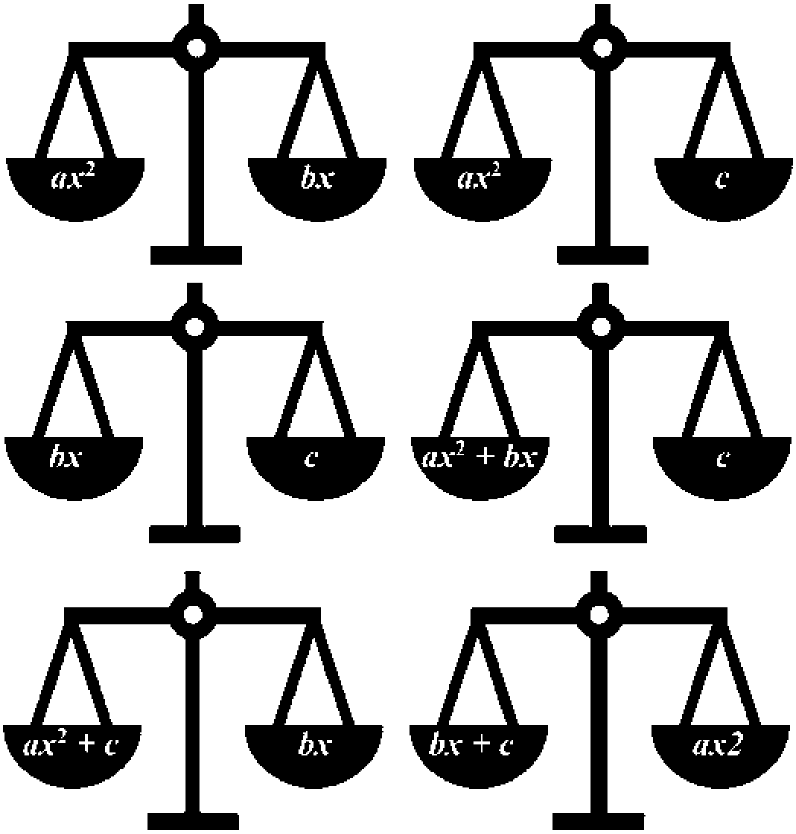


FIGURE 10.7 Al-Khwārizmī's six types of balanced equations

Thus, neither al-Khwārizmī in the ninth century nor Leonardo Pisano, who introduced Hindu Arabic mathematics to Europe in the thirteenth century leveraged India's ideas that would lead to solutions of quadratic equations in forms such as $\boldsymbol{ax^2 + bx + c = 0}$ or $\boldsymbol{ax^2 - bx = {}^{-}c}$. If al-Khwārizmī were to solve for $\boldsymbol{x}$ in $\boldsymbol{x^2 + 2x = 15}$, he would use his rhetorical formula for $\boldsymbol{x^2 + bx = c}$, which today, might be written as shown below:

$$\sqrt{[(b/2)^2 + c]} - b/2$$

Solving for $\boldsymbol{x}$ in $\boldsymbol{x^2 + 2x = 15}$ with $\boldsymbol{b = 2}$ and $\boldsymbol{c = 15}$ is as follows. Two divided by two, is one, which squared, remains one. Add 15 to one and you get 16. Then take the square root of 16, which is four. Then from four, subtract two divided by two, which is one, and you arrive at $\boldsymbol{x = 3}$.

Al-Khwārizmī primarily dealt with unknowns, $\boldsymbol{x}$ (roots or sides of squares), their squares, and rational positive numbers. Had he been exposed to Brahmagupta's negative numbers and zero, he might have solved $\boldsymbol{x^2 + 2x - 15 = 0}$. Today, we look for the factors of $\boldsymbol{x^2}$ and the factors of $\boldsymbol{{}^{-}15}$. The former are $\boldsymbol{x}$

and x while the latter are $^{-}3$ and $^{+}5$ as they sum to 2, the desired coefficient of the middle term, $2x$. So to solve for x in $x^2 + 2x - 15 = 0$ we arrive at $(x + 5)(x - 3) = 0$. As Brahmagupta gave the rule any number multiplied by zero equals zero, either $(x + 5)$ or $(x - 3)$ or both, must equal 0. Thus, the equation $x^2 + 2x - 15 = 0$ is solved with $x = 3$ and $x = -5$. Using these values in the equation confirms $3^2 + 2(3) - 15 = 0$ as $9 + 6 - 15 = 0$, and $(-5)^2 + 2(^{-}5) - 15 = 0$ as $25 - 10 - 15 = 0$. Al-Khwārizmī's approach generated the positive root, 3, yet not the negative root, $^{-}5$. Despite writing 200 years after Brahmagupta, al-Khwārizmī's approach did not feature either zero or more tellingly, negatives, since they remained hidden in India within Brahmagupta's 'lost' definition of zero.

8.4 *On al-Uqlīdisī's Absence of Brahmagupta's Zero*

The oldest extant Arabic text on Indian arithmetic is *Kitāb al-fuṣūl fī al-ḥisāb al Hindī* (The Book of Elements on Indian Arithmetic) by al-Uqlīdisī (*c.*920–980 CE). Written in 952 CE in a textual form without numerals, Chapter 1 is titled *Justification of the Hindi* (*Arithmetic*) *and Its Whys and Hows*. It begins as follows (Saidan, 1978, p. 186):

> Here we state justifications of Hindi (arithmetic) and queries about its whys and hows; for many of the people of this craft ask saying: why and how. To every question that is asked there is an answer and if it is hit, he who asks is satisfied.
>
> One question is: Why are the Hindi letters nine, no more, no less? We say: Because the beginning of numbers from which they start is one and the last unit we pronounce is nine. Thus when we say units we mean (something) between one and nine; after that units are over, and ten comes out like one and takes its form. We add up ten to ten until we reach 90 which conforms with nine. Tens are now over and we say one hundred, coming back to one, and going up to 9. Thus we see that all places start with one and end with nine. That is why they are made nine.

And also:

> If it is said: Why do we say: units, tens, hundreds, thousands? We say: These are places. Upon them lies all the principle of Hindi (arithmetic). The first place is that of units; it may have from one to 9, and these are units only. Next comes the second place, which is the place of tens; it may have from ten to 90 and nothing else. Similarly for the third place which is the place of hundreds, and for the thousands place. No place has more

> than 9; it may have from one to 9, and thereafter we move to another place which may have the same thing again. Thus units, tens, hundreds, and thousands are repeated.

Al-Uqlīdisī ended Chapter 1 on India's number system with *So much for the nine letters*. In Chapter 4, a section titled 'Questions on Multiplication' includes the following:

> If it is said: Why is zero [multiplied] by zero equal to zero and zero by any letter zero? We say that by multiplying zero by zero the aim is only to occupy the place; the same applies for multiplying the letter by zero. We multiply the letter by zero only once, the first time, by the first letter in the upper, to occupy the place, and tell that there is a place and that it is empty.

Thus, it can be seen that al-Uqlīdisī, writing more than 300 years after Brahmagupta, clearly saw zero as being separate to the nine Indian letters (numerals) and that its purpose was to denote an empty place. Notably, al-Uqlīdisī's full name is Abū al-Ḥasan Aḥmed ibn Ibrāhīm ***al-Uqlīdisī***, which means Abū al-Ḥasan Aḥmed ibn Ibrāhīm, ***the Euclidist***. The name of the man who wrote the definitive book on Hindu arithmetic was derived from his advocacy and translations of Euclid's geometry, in which a number zero did not exist. So, by 953 CE, zero may not have been considered a number outside India, yet at least the unit one in the Arabic speaking world was.

8.5 *On Kūshyār ibn Labbān's Absence of Brahmagupta's Zero*

The oldest extant Arabic text that features Hindu numerals is that of Iranian mathematician, Kūshyār ibn Labbān (971–1029 CE). His *Kitáb fī usūl hisāb al-Hind* (Principles of Hindu Reckoning) was written around 1000 CE. Once again, zero was used to fill an empty space where no numbers exist (Levey & Petruck, 1965, pp. 44–46):

> Before proceeding with these principles, it is essential to have a knowledge of the symbols of the nine numerals and the place order of [any] one of them with respect to the others and the increase of [any] one [compared] to the others, and the lessening of [any] one of them compared to the others.

Then, in a section headed 'On the understanding of the symbols of the nine numerals', we read:

> In the place position where there is no number, a zero is placed as a substitute for that missing number. In the case of the ten a cipher is made to precede it in the place position of the units. Likewise the hundred is preceded by 2 zeros in the place position of the units and tens.

India's view of zero as a number defined by an equal prevalence of negative and positive numbers can be contrasted with the Arabic view (Oaks, 2018):

> I have read a few dozen medieval Arabic books on arithmetic and algebra, and there is no hint of negative numbers in any of them. Zero, too, was not regarded to be a number but was merely the placeholder for an empty place in the representation of a number in Arabic (Indian) notation. All numbers in Arabic arithmetic were positive.

Elsewhere we read, 'Like in medieval Europe, negative numbers and zero and were not acknowledged in Arabic mathematics' (Oaks, 2011, p. 2).

9 On the Absence of Brahmagupta's Zero in Europe

9.1 *Leonardo Pisano and the Liber Abaci*

Leonardo Pisano's influential 1228 CE *Liber Abaci*, (Book of Calculation) begins: (Boncompagni, 1857, p. 2)

> Novem figure indorum he sunt 9 8 7 6 5 4 3 2 1. Cum his itaque novem figuris, et cum hoc signo 0, quod arabice zephirum appellatur, scribitur quilibet numerus, ut inferius demonstratur.

This translates as (Sigler, 2002, p. 17): 'The nine Indian figures are: 9 8 7 6 5 4 3 2 1. With these nine figures, and with the sign 0 which the Arabs call *zephir* any number is written, as is demonstrated below.'

Notably, by *any number*, Pisano means any positive number. It is no surprise, we read (Sigler, 2002, p. 6). 'The zero, or zephir as Leonardo calls it, counts for nothing and serves as a place holder.' Exactly six centuries after Brahmagupta gifted the world the elements of modern arithmetic, Pisano's *Liber Abaci* neither documented the laws of sign for positive and negative, nor presented the laws of using zero as a number in the four arithmetical operations.

Pisano is feted for having helped introduce Hindu mathematics to Europe. He did, yet like al-Khwārizmī, also credited with transmitting India's mathematics 400 years earlier, Brahmagupta's symmetric definition of zero as a sum of opposing quantities was long lost in transit, if it even ever left India. On the treatment of equations, as Brahmagupta implied, if the product of two or more factors is zero, then at least one of those factors is zero. Some 200 years later, around 820 CE, al-Khwārizmī missed the idea of 'balancing with zero' to solve equations. Pisano also missed this insight and used the same six standard forms of linear and quadratic equations developed by al-Khwārizmī. Had Brahmagupta's negative numbers and zero been embraced westward, modern equations would most likely have evolved sooner.

As many may remember (or wish to forget), the equation $\boldsymbol{x^2 + 2x - 15 = 0}$ can also be solved with the quadratic formula for determining roots or x-intercepts.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Solving for $\boldsymbol{a} = 1$, $\boldsymbol{b} = 2$ and $\boldsymbol{c} = {}^{-}15$ in $\boldsymbol{x^2 + 2x - 15 = 0}$ is as follows:

$$\frac{-2 \pm \sqrt{2^2 - 4(1)(-15)}}{2} \text{ which is } \frac{-2 \pm \sqrt{2^2 - (-60)}}{2} \text{ then } \frac{-2 \pm \sqrt{64}}{2} \text{ and } \frac{-2 \pm 8}{2}$$

so $\boldsymbol{x = 3}$ and $\boldsymbol{x = {}^{-}5}$.

With $\boldsymbol{x^2 + 2x - 15 = 0}$, India's zero multiplication rule means either factor $(\boldsymbol{x + 5})$ or $(\boldsymbol{x - 3})$ must equal zero. This is simpler than both the formula of al-Khwārizmī for $\boldsymbol{x^2 + 2x = 15}$ which is $\sqrt{[(\boldsymbol{b}/2)^2 + \boldsymbol{c}]} - \boldsymbol{b}/2$ and the quadratic formula above.

So, it took around 1,000 years for Brahmagupta's zero to truly be given the same algebraic status as a number in its own right, that combinations of other numbers could equal. Notably, the first to apply the zero definition of Brahmagupta to arrive at the general formula for quadratic equations was India's Śrīdhara (*c.*870–*c.*930 CE).

9.2 *How Positives Became Negative and India's Symmetric Zero Was Assumed Understood*

You may not have noticed that equal and opposite negatives opposing positive quantities have been missing in books on the evolution of Western mathematics. The reason is we often read about negative numbers (that are not) in the writings of mathematicians throughout history. The Greek Diophantus of Alexandria (circa 250 CE), wrote the following in his *Arithmetica*: Λεῖψις ἐπι λεῖψιν πολλαπλασιασθεἰσα ποιεῖ ὕπαρξιν, λεῖψις δέ επί ὕπαρξιν ποιεί λεῖψιν (Heath, 1910, p. 130). In Latin, this was correctly translated as (Bachet de Méziriac, 1621, p. 9) *Minus per minus multiplicatum, producit Plus. At minus per plus multiplicatum, producit minus*. Heath gave this translation of Diophantus: 'A minus multiplied by a minus makes a plus; a minus multiplied by a plus makes a minus.'

Importantly, on the nature of $(\boldsymbol{a} - \boldsymbol{b}) \times (\boldsymbol{c} - \boldsymbol{d})$, in Heath we read: 'To be emphasized is the fact that in Diophantus the fundamental algebraic conception of negative numbers is "wanting".' *In 2x – 10 he avoids as absurd all cases where 2x < 10*. In $(\boldsymbol{a} - \boldsymbol{b}) \times (\boldsymbol{c} - \boldsymbol{d})$, the terms $\boldsymbol{b}$ and $\boldsymbol{d}$ are not negative terms. Despite the fact $\boldsymbol{b}$ and $\boldsymbol{d}$ are positive terms being subtracted, we read comments such as 'Diophantus formulates for relative numbers the following rule of signs: a negative multiplied by a negative yields a positive, whereas a negative by a positive yields a negative' (Bashmakova & Silverman, 1997, p. 6). Diophantus did not write about positive and negative numbers related by zero and neither did al-Khwārizmī, yet you may not have noticed.

9.3 *How Explanations of* al-jabr *Misled Educators*

As mentioned, al-Khwārizmī eliminated subtracted terms and only confronted positive terms on opposing sides to 'balance' his equations. Yet through modern eyes, subtracted terms often magically morph into negative terms, which implies zero mediates the positive and negative. Al-Khwārizmī's algebraic equations never contained negative terms, yet we read:

> In mathematical language, the verb [jabr] means … when applied to equations, to transpose negative quantities to the opposite side by changing their signs. The negative quantity thus removed. (Rosen, 1831, p. 178)

and

> The usual meaning of jabr in mathematical treatises is: adding equal terms to both sides of an equation in order to eliminate negative terms. (van der Waerden, 1985, p. 4)

and

> Al-jabr means 'restoration' or 'completion', that is, removing negative terms, by transposing them to the other side of the equation to make them positive. (Devlin, 2012, p. 53)

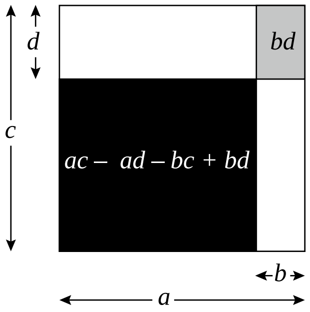


FIGURE 10.8
The geometric instantiation of '*minus* × *minus* = *plus*' given false equivalence to *'negative* × *negative* = *positive'*

Positive numbers being subtracted are not the same thing as negative numbers, yet they are often conflated. In the equation $a - b = c$ (where $b < a$) the term b is not a negative number. Mathematicians before and after Brahmagupta, such as Diophantus and al-Khwārizmī, have given geometric explanations for $(a - b) \times (c - d)$ provided $a > b$ and $c > d$. Yet, all the terms were positive, with b always less than a and d always less than c. Today teachers might discuss the expansion of $(a - b) \times (c - d)$ which becomes $ac - ad - bc + bd$ to say we define *negative multiplied by negative as positive* in order to preserve the distributive property of multiplication. Looking more closely at $(a - b) \times (c - d)$ the term bd is a positive term being added back after having been subtracted once too many times as shown in Figure 10.8.

With $(a - b) \times (c - d)$ in the form $ac - ad - bc + bd$, from ac, we subtract $a \times d$, (the white horizontal strip from left to right) and subtract $b \times c$, (white vertical strip from bottom to top) to get $ac - ad - bc$. Yet we *twice* subtracted bd (the gray shaded area resulting from overlapping white strips), so we add back *bd* in the corner once, to get $ac - ad - bc + \underline{bd}$. With $(10 - 2) \times (10 - 3)$ as the example for $(a - b) \times (c - d)$, we know the answer is 8×7, which is 56. So, by distribution over subtraction we get 100 (i.e., ac) – 30 (i.e., ad) – 20 (i.e., bc). We have arrived at $100 - 30 - 20$ which is 50, yet we must make an adjustment as we twice subtracted the gray area bd. Therefore, we add back bd once to arrive at $ac - ad - bc + \underline{bd}$ and get $100 - 30 - 20 + (2 \times 3) = 56$. Notably, every term in this explanation is positive, regardless of whether it is added or subtracted. Subtracted terms have for so long been anachronistically described as negative terms, that many people have thought negative algebraic terms were a feature of al-Khwārizmī's equations. Similarly, we have long seen discussions about positive and negative exponents with a^{+b} and a^{-b} where a is the base and b is the exponent. Yet the signs of the exponent are misleading. They might be better written $a^{\times b}$ and $a^{\div b}$ as the exponents are a count of the number of times 1 is either multiplied by a or divided by a. For example, 2^{+3} more clearly means $2^{\times 3}$ as it becomes $1 \times 2 \times 2 \times 2$ which is 8 and 2^{-3} more clearly means

$2^{\div 3}$ as it becomes $1 \div 2 \div 2 \div 2$ which is 1/8. The problematic $\boldsymbol{a}^{\mathbf{0}}$ (which often gets dumped into the index laws) then becomes simple. With $\boldsymbol{a}^{\mathbf{0}}$ the number of times 1 is either multiplied by $\boldsymbol{a}$ or divided by $\boldsymbol{a}$ is zero, so 1 remains unchanged, thus $\boldsymbol{a}^{\mathbf{0}} = 1$.

10 How Brahmagupta's Zero Could Have Helped Diophantus and Descartes

The author's analogy, inspired by fond memories of playing with cubic bricks, is just as a wall is composed of a number of bricks, we do not consider a single brick a wall. Yet where might a cubic mud brick have come from? The answer is, of course, a hole in the ground. Before the brick and hole were created, we had ground-level zero, treating height above and below ground level as an implicit vertically aligned number line. Consistent with the physical law of conservation of matter, for every brick made another hole is made as zero is split and rearranged into opposite quantities in the spirit of Brahmagupta's definition.

Diophantus called the equation $4 = 4\boldsymbol{x} + 20$ absurd as it would result in a negative value for $\boldsymbol{x}$, which he thought was impossible. Yet, what if Diophantus had played with Brahmagupta as a child? Brahmagupta might have said he had four bricks, which was the same as Diophantus' 20 bricks combined with four things. After a bit of fun, they would have realized the four mystery things were holes, each four bricks deep. In the game a brick would be a positive unit and a hole a negative unit, while obeying both India's laws of mathematics and the basic laws of physics. Having imagined taking Brahmagupta back to the third century to play with Diophantus, what if Brahmagupta had played with René Descartes in the seventeenth century?

Like others in the West before him, Descartes only considered line segments to be positive. After all, how can you travel a distance less than nothing? Importantly, borrowing from an ancient idea of Euclid, Descartes came up with a model of multiplication involving line segments where the product was another line, rather than an area (Descartes, 1637, p. 298). A modernized comparison of the area model of multiplication with Descartes' idea of using similar triangles is shown in Figure 10.9 (Crabtree, 2017b, pp. 94–96).

Descartes never extended his line segments backward onto the opposite side of the origin (zero), yet Brahmagupta might have suggested he do just that. After all, Brahmagupta surely knew if from his origin he walked 100 steps south then 100 steps north, the net distance he would have traveled from his starting point to his endpoint would be zero. Had Brahmagupta told Descartes negative lines were simply equal and opposite to positive lines on the other

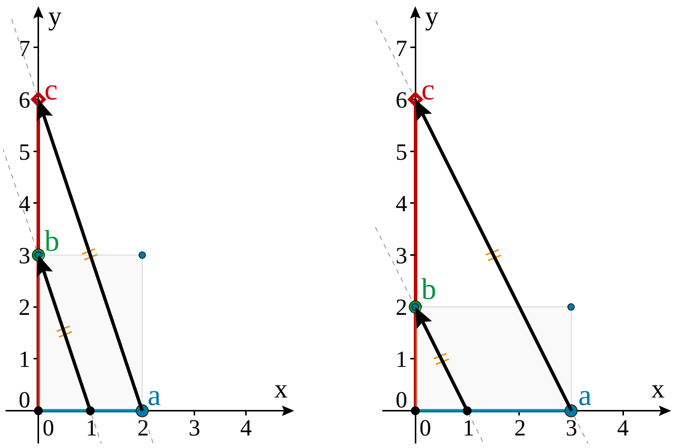


FIGURE 10.9
Standard area models for 2×3 and 3×2 alongside modernized versions of Descartes' original proportional approach

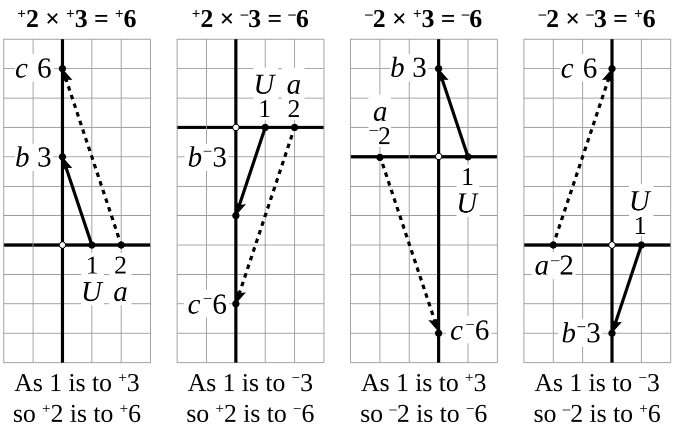


FIGURE 10.10
Instantiations that might have emerged had Brahmagupta met Descartes

side of zero, children might now be taught laws of sign via similar triangles (Figure 10.10).

11 Rebuilding Elementary Mathematics from Zero

From the time of the ancient Greeks, 'number theory' has been developed on the natural numbers where N = {1, 2, 3, …}. For example, the first axiom in the highly influential late nineteenth century Dedekind/Peano axioms stipulated 1 to not be the successor of any number, implying zero was not a number. Joseph Peano eventually included zero in his axioms (Peano, 1902, p. 8) yet it had little effect on the academic status of zero as we continue to read 'there exists no number whose successor is 1' (Landau, 1966: 2). For all the time scholars have invested in number theory, there has been no pedagogical benefit in lower-level classrooms. Bad English ideas from centuries ago remain just as bad in classrooms today. For example, in 1685, the English mathematician John Wallis

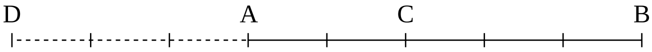


FIGURE 10.11 John Wallis's diagram with which he said a movement from A to D was less than no move at all

[Wallis, 1685] drew a diagram in which a movement west of an origin at A with a magnitude of three from A to D was described as 'less than nothing', which is why three negatives are still said to be less than 0 negatives today (Figure 10.11).

12 Teaching 2 – 5

So, how might a child understand 2 – 5? Being familiar with the idea of a bucket and spade, children can understand that to make one brick you must first dig one hole. Now, in the role of a brick seller, what might happen if a child had only two bricks for sale yet five were needed by a customer? The child would simply dig three more holes to make three more bricks. The child now has five bricks and three holes. Once the five bricks are taken away, the three holes remain. Simply assign the idea of positives to bricks and negatives to holes and the pedagogy of 2 – 5 is self-evident. The absence of bricks and holes can represent a nothing zero while the presence of equal numbers of brick and equal numbers of holes of equal dimension can represent an Indian zero.

In a physical sense, ${}^{+}2 - {}^{-}5$ cannot be resolved as you only have positive quantities and so cannot subtract negative quantities. A solution is to add a Brahmaguptan zero-pair in the form of ${}^{+}5 + {}^{-}5$. The equation then becomes ${}^{+}2 + {}^{+}5 + {}^{-}5 - {}^{-}5$ which is ${}^{+}7$. Alas, ${}^{+}2 - {}^{-}5$ and ${}^{-}5 - {}^{+}2$ is taught five years after 5 – 2 because India's zero definition as a sum of equal and opposite quantities, empirically consistent with science, failed to be transmitted to the West.

So, if God made the integers, the devil is in the detail. We cannot reclaim lost progress. Yet will the world be prepared to rebuild its curriculums upon Indian foundations that featured both zero and negative quantities rather than Greek foundations that did not? Only time will tell. In the meantime, further pedagogical research based on visual instantiations of Brahmagupta's writings on zero, negatives and positives consistent with physical laws (Crabtree, 2018) is suggested.

References

Bachet, Claude. (1621). *Diophanti Alexandrini Arithmeticorum Libri Sex*. Lutetiae, Parisorum.

Barrow, John D. (2001). *The Book of Nothing*. Vintage, London.

Bashmakova, Izabella G., and Joseph H. Silverman. (1997). *Diophantus and Diophantine Equations*. Mathematical Association of America, Washington, DC.

Boncompagni, Baldassarre. (1857). *Il Liber Abbaci Di Leonardo Pisano*. Fiorentina, Badia.

Borowski, Ephraim, and Borwein, Jonathan. (2012). *Collins Dictionary of Mathematics*. Glasgow, UK: Harper Collins.

Colebrooke, Henry T. (1817). *Algebra with Arithmetic and Mensuration from the Sanskrit of Brahmegupta* (sic) *and Bhascara*. Murray, London.

Crabtree, Jonathan J. (2016). The Lost Logic of Elementary Mathematics and the Haberdasher Who Kidnapped Kaizen. *Proceedings of the Mathematical Association of Victoria Annual Conference 53*, Melbourne, Australia. Retrieved March 2019 from www.mav.vic.edu.au/files/2016/2016_MAV_conference_proceedings.pdf.

Crabtree, Jonathan J. (2017a). Fun in Podo's Paddock: New Ways of Teaching Children the Laws of Sign for Multiplication and Division (and more!). *The Mathematics Education for the Future Project: Proceedings of the 14th International Conference Challenges in Mathematics Education for the Next Decade*, Balatonfüred, Hungary. Retrieved March 2019 from www.bit.ly/TeachingLawsOfSign.

Crabtree, Jonathan J. (2017b). How mathematics teachers can explain multiplication and division in the manner of René Descartes and Isaac Newton. *Proceedings of the 26th Biennial Conference of the Australian Association of Mathematics Teachers* 90–98, The Australian Association of Mathematics Teachers (AAMT) Inc., Adelaide, South Australia.

Crabtree, Jonathan J. (2018). *The Relevance of Indian Mathematics*. Presentation for the Department of Mathematics, Jadavpur University, Kolkata, India. Retrieved March 2019 from www.bit.ly/NewMaths.

Crossley, John N., and Henry, Alan S. (1990). *Thus Spake Al-Ḵẖwārizmī: A Translation of the Text of Cambridge University Library Ms. Ii. Vi. 5*. Historia Mathematica.

Datta, Bibhutibhusan, and Singh, Avadhesh N. (1962). *History of Hindu Mathematics*. Asia Publishing House, Bombay.

Descartes, R. (1637). *La Géométrie in Discours de la Méthode*. I. Maire: Leyde.

Devlin, Keith. (2012). *The Man of Numbers*. Bloomsbury Publishing, London.

Dvivedin M. Sudhakāra. (1902). *Brāhmasphuṭasiddhānta and Dhyānagrahapadeṣādhyāya*. Printed at the Medical Hall Press, Benares.

Farrar, John. (1818). *An Elementary Treatise on Arithmetic, Taken from the Arithmetic of S. F. Lacroix*, Printed by Hilliard & Metcalf, at the University Press, Cambridge, New England.

Heath, Thomas, L. and Euclid. (1908). *The Thirteen Books of Euclid's Elements, Vol II*. Cambridge: At the University Press.

Heath, Thomas L. (1910). *Diophantus of Alexandria: A Study in the History of Greek Algebra*. Cambridge University Press.

Ifrah, Georges. (2000). *The Universal History of Numbers*. John Wiley, New York.

Joseph, George G. (2016). *Indian Mathematics: Engaging with the World, from Ancient to Modern Times*. World Scientific, London.

Landau, Edmund G. (1966). *Foundations of Analysis: The Arithmetic of Whole, Rational, Irrational and Complex Numbers*. Chelsea Publishing, New York.

Levey, Martin and Petruck, Marvin. (1965). *Principles of Hindu Reckoning*. University of Wisconsin Press, Madison.

Martzloff, J.-C. (2006). *A history of Chinese mathematics*. New York: Springer.

Oaks, Jeffrey A. (2018). Personal communication with Dr Jeffrey Oaks, Professor of Mathematics on Medieval Arabic algebra and the mathematics of Greece and medieval Europe.

Oaks, Jeffrey A. (2011). *Al-Khayyām's Scientific Revision of Algebra*.

Oughtred, W. (1631). *Arithmeticæ in numeris et speciebus institutio: quæ tum logisticæ, tum analyticæ, atque adeo totius mathematicæ, quasi clavis est, etc*. Apud T. Harperum: Londini.

Peano, Giuseppe. (1902). *Aritmetica Generale E Algebra Elementare*. (General Arithmetic and Elementary Algebra) Ditta G. B. and Paravia C., Torino.

Pingree, David. (1981). *Jyotiḥśāstra: Astral and Mathematical Literature*. Wiesbaden: Harrassowitz.

Plofker, Kim. (2009). *Mathematics in India*. Princeton University Press, Princeton.

Prakash, Satya. (1968). *A Critical Study of Brahmagupta and His Works*. Indian Institute. of Astronomical and Sanskrit Research, New Delhi.

Raju, Chandra K. (2007). *Cultural Foundations of Mathematics: The Nature of Mathematical Proof and the Transmission of the Calculus from India to Europe in the 16th c. CE*. Pearson Longman, Project of history of Indian Science, philosophy and culture, Delhi.

Rashed, Roshdi. (2009). *The Beginnings of Algebra*. Saqi, London.

Rorres, Chris. (2003). *Infinite Secrets*, www.pbs.org/wgbh/nova/transcripts/3010_archimed.html.

Rosen, Fred. A. (1831). *The Algebra of Mohammed Ben Musa*. J. L. Cox, London.

Saidan, Ahmad S. (1978). *The Arithmetic of Al-Uqlídisí: The Story of Hindu-Arabic Arithmetic As Told in Kitab Al-Fusul Fi Al-Hisab Al-Hindi*. Reidel, Dordrecht.

Shen, K., Liu, H., Crossley, J. N., and Lun, A. W.-C. (1999). *The Nine Chapters on the Mathematical Art: Companion and Commentary*. Oxford: Oxford University Press.

Sigler, Laurence. (2002). *Fibonacci's Liber Abaci: a Translation into Modern English of Leonardo Pisano's Book of Calculation*. Springer, New York.

Singh, S. (2021). *Chasing Rabbits: A Curious Guide to a Lifetime of Mathematical Wellness*. Impress, San Diego.

Strachey, Edward. (1813). *Bija Ganita: Or the Algebra of the Hindus*. Glendinning, London.

Taleb, N. N. (2007). *The Black Swan: The Impact of the Highly Improbable*. London: A. Lane.

Taylor, John and Bhāskarāchārya. (1816). *Lilawati: Or, a Treatise on Arithmetic and Geometry*. Printed at the Courier Press, by S. Rans, Bombay.

Van der Waerden, Bartel, L. (1985). *A History of Algebra, from Al-Khwārizmī to Emmy Noether*. Springer, Berlin.

Wallis, John. (1685). *A Treatise of Algebra, both Historical and Practical*. Printed by John Playford, for Richard Davis, Bookseller, in the University of Oxford.