Notes on: Brahmagupta's definition of zero failing to be transmitted to Europe via the Arabic world

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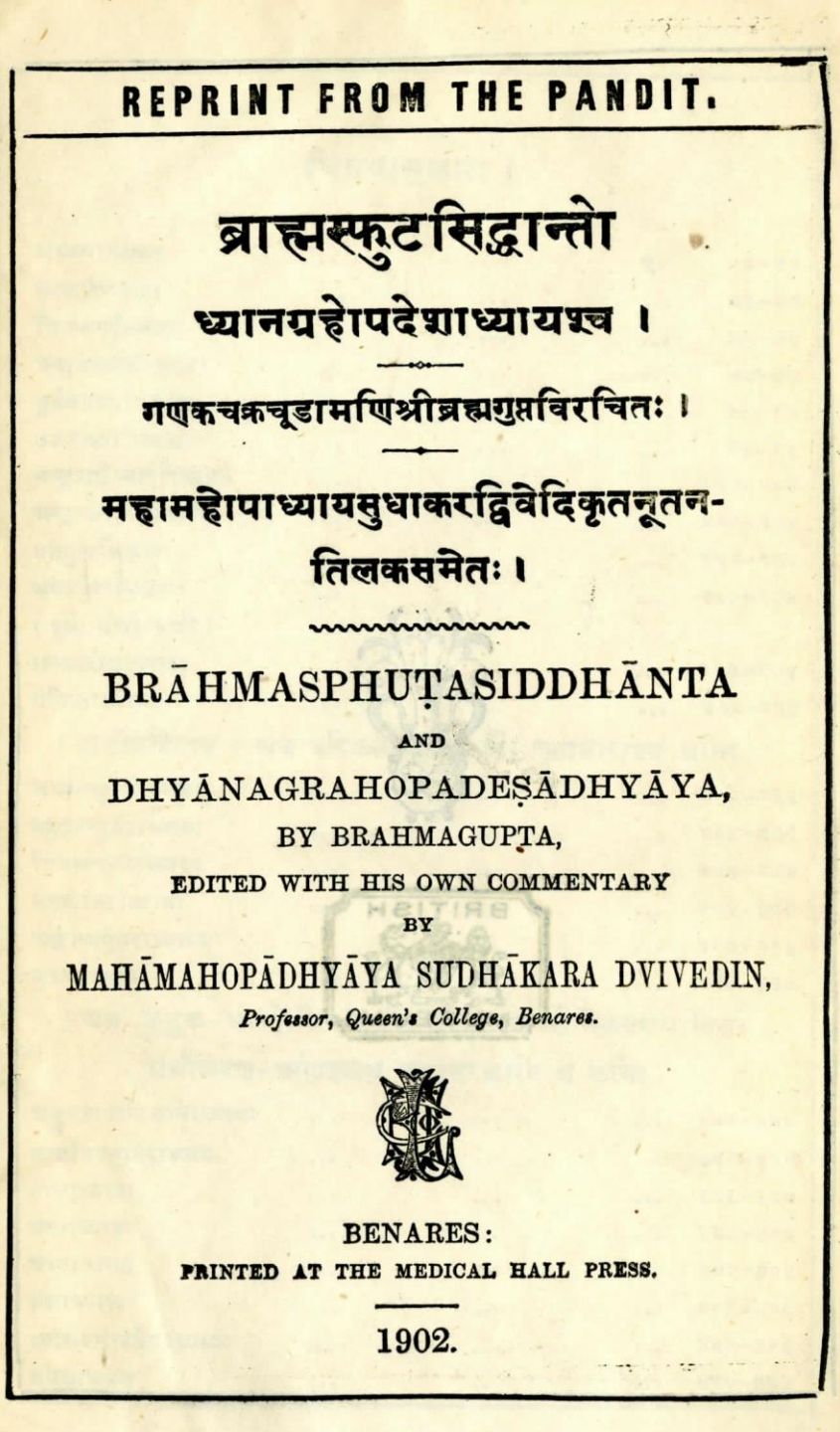
“Elementary maths foundations concern counts, measures and relationships between quantities that reveal predictable patterns”

And counts of absolute value binary operations

You are familiar with Brahmagupta's 7th C. sign laws* for positives, negatives and zero.

* 628 CE Brāhmasphuṭa Siddhānta 18: 30-35



REPRINT FROM THE PANDIT.

ब्राह्मस्फुटसिद्धान्तो

ध्यानग्रहोपदेशाध्यायश्च ।

गणकचक्रचूडामणिश्रीब्रह्मगुप्तविरचितः ।

महामहोपाध्यायसुधाकरद्विवेदिकृतनूतन-
तिलकसमेतः ।

BRĀHMASPHUṬASIDDHĀNTA
AND
DHYĀNAGRAHOPADEṢĀDHYĀYA,
BY BRAHMAGUPṬA,
EDITED WITH HIS OWN COMMENTARY
BY
MAHĀMAHOPĀDHYĀYA SUDHĀKARA DVIVEDIN,
Professor, Queen's College, Benares.

BENARES:
PRINTED AT THE MEDICAL HALL PRESS.

1902.

P.1

अथ धनर्णशून्यानां सङ्कलनम् ।

धनयोर्धनमृणमृणयो-
र्धनर्णयोरन्तरं समैक्यं खम् ।
ऋणमैक्यं च धनमृणध-
नशून्ययोः शून्ययोः शून्यम् ॥ ३० ॥ (३१)

धनयोरैक्यं धनमृणयोरैक्यमृणं भवति । धनर्णयोरन्तरमेवैक्यं भवति । समयोर्धनर्णयोरैक्यं खं शून्यं भवति । ऋणशून्ययोरैक्यमृणं धनशून्ययोरैक्यं धनं शून्ययोरैक्यं च शून्यं भवति ।

अत्रोपपत्त्यर्थं मन्मुद्रिता भास्करबीजटिप्पणी द्रष्टव्या ॥ ३० ॥

इदानीं व्यवकलनमाह ।

ऊनमधिकाद्विशोध्यं धनं धनादृणमृणादधिकमूनात् ।
व्यस्तं तदन्तरं स्यादृणं धनं धनमृणं भवति ॥ ३१ ॥ (३२)
शून्यविहीनमृणमृणं धनं धनं भवति शून्यमाकाशम् ।
शोध्यं यदा धनमृणादृणं धनाद्वा तदा क्षेप्यम् ॥ ३२ ॥ (३३)

अधिकादूनादूनं धनं विशोध्यं शेषं धनं भवति । अधिकादृणादूनमृणं विशोध्यं शेषमृणं भवति । ऊनाद्धनादधिकं धनं वोनादृणादधिकमृणं विशोध्यं तदा तदन्तरं व्यस्तं विपरीतं स्यात् । अर्थादधिकं धनं विशोध्यं तदा शेषमृणं भवति । अधिकमृणं विशोध्यं तदा शेषं धनं भवति । कथं विपरीतं भवतीत्याह । ऋणं धनं भवति धनं चर्णं भवतीति । चेदृणं शून्यविहीनं शून्येन विहीनं तदा ऋणं धनं च शून्यविहीनं धनं शून्यं च शून्यविहीनमाकाशं शून्यं भवति । यदि ऋणादूनं शोध्यं वा धनादृणं शोध्यं तदा क्षेप्यमर्थात् तदा तयोर्योग एवान्तरं भवतीति ।

अत्रोपपत्त्यर्थं मन्मुद्रिता भास्करबीजटिप्पणी विलोक्या ॥ ३१-३२ ॥

इदानीं गुणने करणसूत्रम् ।

ऋणमृणधनयोर्घातो धनमृणयोर्धनवधो धनं भवति ।
शून्यर्णयोः खधनयोः खशून्ययोर्वा वधः शून्यम् ॥ ३३ ॥ (३४)

ऋणधनयोर्घात ऋणं भवति । ऋणयोर्वधो धनवधो धनयोर्वधश्च धनं भवति । शून्यर्णयोः खधनयोः शून्यधनयोर्वा खशून्ययोश्च वधः शून्यं भवति ॥ ३३ ॥

इदानीं भागहारे करणसूत्रं वृत्तद्वयम् ।

धनभक्तं धनमृणहृतमृणं धनं भवति खं खभक्तं खम् ।
भक्तमृणेन धनमृणं धनेन हृतमृणमृणं भवति ॥ ३४ ॥ (३५)
खोद्धृतमृणं धनं वा तच्छेदं खमृणधनविभक्तं वा ।
ऋणधनयोर्वर्गः स्वं खं खस्य पदं कृतिर्यत् तत् ॥ ३५ ॥ (३६)

धनं धनभक्तं वा ऋणं ऋणभक्तं फलं धनं भवति । खभक्तं खं फलं खं भवति । ऋणेन धनं भक्तं फलमृणं स्यात् । धनेन ऋणं हृतं फलमृणं भवति । ऋणं वा धनं खेनोद्धृतं तच्छेदं तस्य शून्यस्य छेदो यस्मिन् ऋणे वा धने तच्छेदं भवति । एवं खं शून्यमृणधनविभक्तं (शून्यं) वा तच्छेदं भवति । फलं शून्यं भवति वा शून्यं तद्धरं स्यादित्यर्थः । ऋणधनयोर्वर्गः स्वं भवति । खस्य वर्गः खं भवति । तदेव वर्गस्य पदं भवति यत्कृतिः स एव वर्गो भवेदिति । भास्करबीजेऽप्येतदेव सर्वम् । अत्र खभक्तं खमर्थात् $\frac{0}{0}$ इदं सर्वदा शून्यसमं नेत्येतदर्थं चलनकलनं विलोक्यम् ॥ ३४-३५ ॥

इदानीं सङ्क्रमणविषमकर्माह ।

योगोऽन्तरयुतहीनो द्विहृतः सङ्क्रमणमन्तरविभक्तं वा ।
वर्गान्तरमन्तरयुतहीनं द्विहृतं विषमकर्म ॥ ३६ ॥ (३७)

योगो राश्योर्योगोऽन्तरेण राश्यन्तरेण युतो हीनश्च द्विहृतो दलितो राशी स्तः । इदं सङ्क्रमणं नाम गणितम् । वा राश्योर्वर्गान्तरं राश्यन्तरेण विभक्तं फलमन्तरेण युतं हीनं द्विहृतं च राशी स्तः । इदं विष-

From chapter 18 on algebra, 18 simple sutras* of symmetry emerge that agree with basic laws of physics.

* Plus a contentious concept for khahara or division by zero

Brahmagupta's 5 Addition Sutras

धनयोर्धनम्ऋणमृणयोः धनर्णयोरन्तरं समैक्यं खम् ऋणमैक्यं च धनमृणधनशून्ययोः शून्ययोः शून्यम्

AS1 positive plus positive is positive

AS2 negative plus negative is negative

AS3 positive plus negative is the difference between the positive and negative

AS4 when positive and negative are equal the sum is zero

AS5
positive plus zero is positive
negative plus zero is negative
zero plus zero is zero

Brahmagupta's 5 Subtraction Sutras

ऊनमधिकाद्विशोध्यं धनं धनाद्ऋणमृणाद्अधिकमूनात् व्यस्तं तदन्तरं स्याद्ऋणं धनं धनमृणं भवति
शून्यविहीनमृणमृणं धनं धनं भवति शून्यमाकाशम् शोध्यं यदा धनमृणाद्ऋणं धनाद्वा तदा क्षेप्यम

SS1 A smaller **positive** subtracted from a larger **positive** is **positive**.

SS2 A smaller **negative** subtracted from a larger **negative** is **negative**.

SS3 If a larger **negative** or **positive** is to be subtracted from a smaller **negative** or **positive**, the sign of their difference is reversed – **negative** becomes **positive** and **positive** **negative**.

SS4 A **negative** minus **zero** is **negative**,
a **positive** minus **zero** is **positive**,
zero minus **zero** is **zero**.

SS5 When a **positive** is to be subtracted from a **negative** or a **negative** from a **positive**, then it is to be added.

Brahmagupta's 4 Multiplication Sutras

ऋणमृणधनयोर्घातो धनमृणयोः धनवधो धनं भवति
शून्यर्णयोः खधनयोः खशून्ययोर्वा वधः शून्यम्

MS1	The product of a **negative** and a **positive** is **negative**.
MS2	The product of two **negatives** is **positive**.
MS3	The product of two **positives** is **positive**.
MS4	The product of **zero** and a **negative**, of **zero** and a **positive**, or of two **zeros** is **zero**.

Brahmagupta's 4 Division Sutras

धनभक्तं धनम् ऋणहृतमृणं धनं भवति खं खभक्तं खम्
भक्तमृणेन धनमृणं धनेन हृतम् ऋणमृणं भवति
खोद्धृतमृणं धनं वा तच्छेदं खमृणधनविभक्तं वा
ऋणधनयोर्वर्गः स्वं खं खस्य पदं कृतिर्यत् तत्

DS1 A **positive** divided by a **positive** is **positive**.

DS2 A **negative** divided by a **negative** is **positive**.

DS3 A **positive** divided by a **negative** is **negative**.

DS4 A **negative** divided by a **positive** is **negative**.

Despite this symmetric zero-based genesis, a brief cross-cultural review reveals an incomplete understanding of Bharat's zero in the medieval Arabic world.

It appears zero as a place-holder was transmitted via the Arabic world to renaissance Europe.

Yet the role of zero as a number, defined by Brahmagupta as the sum of equal yet opposing positive and negative quantities… was neither grasped in the Middle East nor transmitted to Europe!

Brahmagupta, I wasn't told about your 0 definition or your 18 Sūtras of ZERO, Positives & Negatives!

Recorde did know a subtraction subtracted is like addition yet that's another idea.

Transmission of zero as a placeholder, yet not as defined by Brahmagupta 628 CE.

Those unaware zero was defined as the sum of equal pos. & neg.

- Al-Khwārizmī, Iraq 9th C.
- Traders, Nth Africa 12th C.
- Leonardo Pisano, Italy 13th C.
- Robert Recorde, England 16th C.

So, neither negatives nor zero appear to have been involved in the development of Arabic algebra.

These sutras or rules are correct yet, **21st C. maths** pedagogies remain **disconnected** from post-Vedic **Bharatiya maths**.

This talk reveals meta mathematical disconnects and how they imperceptibly arose during the passage of time and place.

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@jcrabtree

FYI Sanskrit isn't needed to learn #podometic!

Thanks to the help of Sanskritist maths professors (E.g. Dr. Avinash Sathaye, Dr. K. Ramasubramanian etc.) plus others who helped me explore more languages I've done enough analysis & interpretation to rebuild basic maths from ZERO!

888	Greek	ἀριθμὸς ἀριθμὸν πολυπλασιάζειν λέγεται. ὅτ’αν ὅσαι εἰσιν ἐν αὐτῶι μονάδες τοσαυτάκις συντεθῆι ὁ πολλαπλασιαζόμενος καὶ γένηταί τις
950	Arabic	الضرب هو أن يوجد أحد العددين بعدد آحاد العدد الآخر فيكون حصة الواحد من آحاد المضروب هي المضروب فيه بعينه والمجموع هو العدد الحاصل من ضرب العدد
1482	Latin	Numerous per alium multiplicari dicitur, qui totiens sibi coacervatur, quotiens in multiplicante est unitas.
1543	Italian	Quel numero se dice esser multiplicato per un'altro, il quale si e assunato tante volte, quante unita e in lo multiplicante.
1555	German	Ain zal multiplicirt oder meret ain andere / wann die ander / als offt die erst zal ains in jr beschleüßt / genommen vnd zuesamen bracht wirdt. Als 4. multiplicirt oder meret die zal 7. wann die zal 7. vier mal / in ansehen das ains in 4. viermal begriffen ist / genommen vnd zuesamen bracht wirdt.
1565	French	Un nombre, se dict multiplier un autre nombre, quand autant d'unitez, qu'il y a en luy, autant de fois se compose le multiplie, & en naist un autre.
1570	English	A number is sayd to multiply a number, when the number multiplyed, is so oftentimes added to itselfe, as there are in the number multiplying unities : and an other number is produced.
1665	Spanish	Un número se dice multiplicar á otro quando tantas veces estuviere compuesto el que se multiplica, quantas fueren las unidades del multiplicador, y el producto fuere algun número.
1695	Dutch	Een getal segt men een getal te vermeenigvuldigen, als dat soo meenigmaal een saamgeset getal is, dat vermeenigvuldigt word, als ‘er eenheden in de vermeenigvuldigende sijn, en dat ‘er eenig getal voortkomt.
1719	Sanskrit	गुण्याङ्कगुण्काङ्कयोर्घातो गुणनफलं क्षेत्रफलं भवति
1855	Swedish	Ett tal säges multiplicera ett tal, när det sednare talet tages så många gånger, som enheter finnas i det förra, och ett annat tal (produkten) deraf uppkommer
1857	Chinese	乘數者，數有若干倍，即若干為乘數。面數者， 兩數相乘所得，原兩數為其邊。
1865	Hungarian	Szám számot szorozni mondatik, midon a hány egység van benne, annyiszor rakatik a szorzandó, és igy származik szám.
1907	Czech	Pravíme, že číslo číslem se násobí, když násobené (násobenec) tolikrát se složí, kolik v druhém jest jednotek, a nějaké vznikne.
1912	Hebrew	והמספר המנוי במספר אחר הוא המספר הנכפל פעמים אשר מנינם כמנין האחדים אשר במספר השני אשר הוא נמנה בו, כמו שתי פעמים שלש או שתי פעמים עשרה והוא הנקרא מספר שטוח וזו צ ו ר ת ו : : : והמספר הנקבע מהכפל הזה יקרא מספר שטוח
1912	Danish	Et Tal siges at multiplicere et Tal, naar det, som multipliceres, lægges sammen ligesaa mange Gange, som der er Enheder i det første, og et eller andet Tal frembringes.
1949	Russian	Говорят, что число умножает число, когда сколько в нем единиц, столько раз составляется умножаемое и что-то возникает.

Which way is she spinning?

R

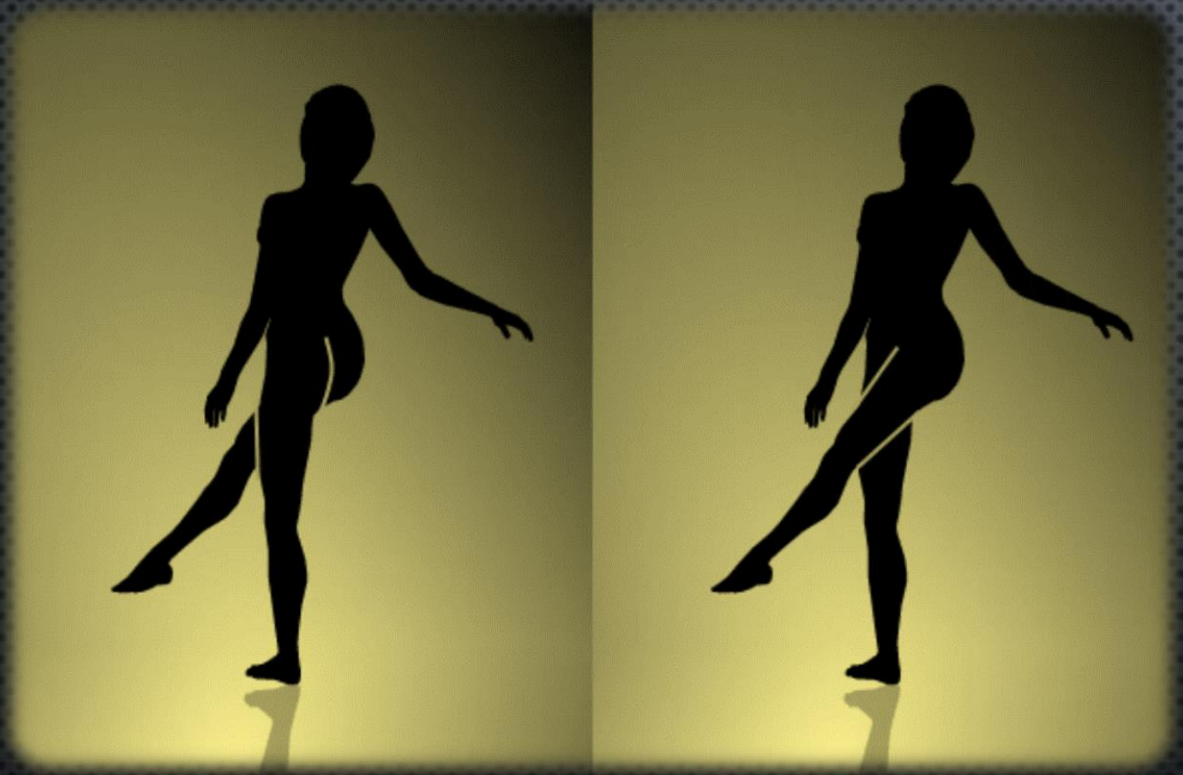
Clockwise

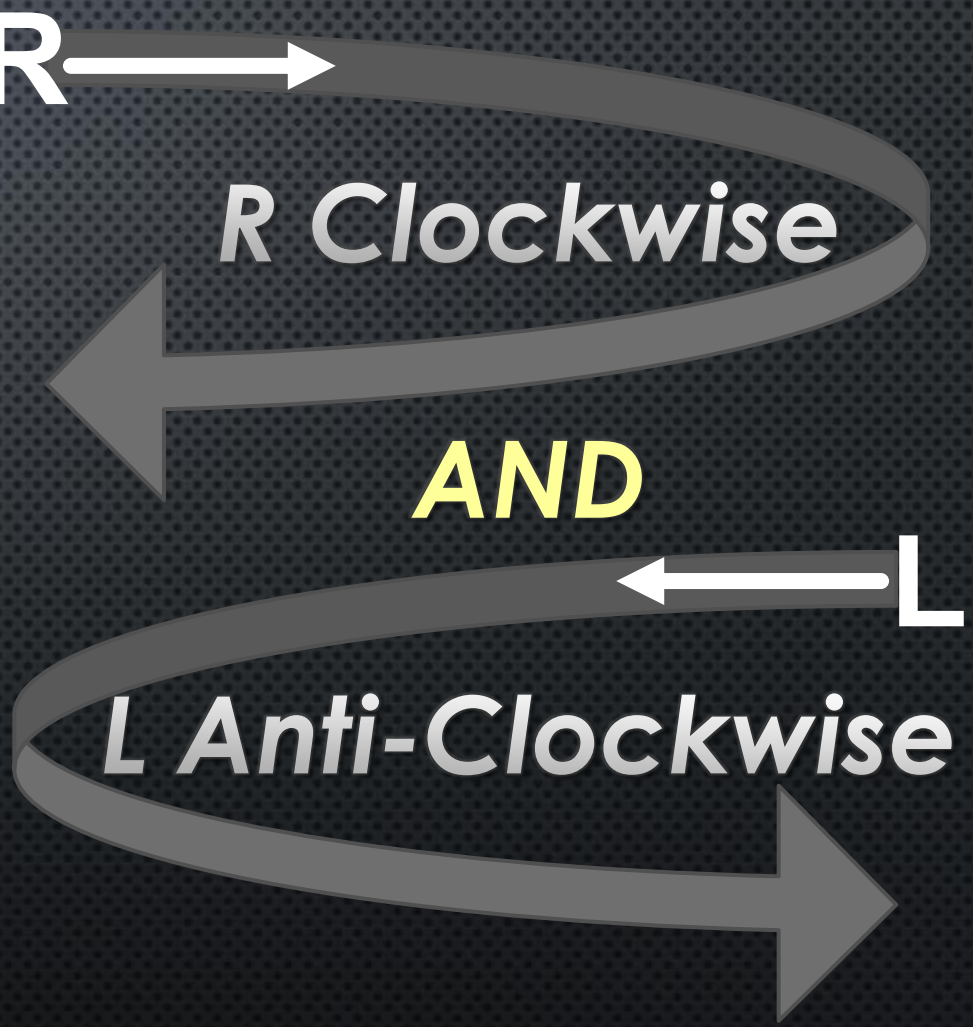
or

L

Anti-Clockwise

LIKE MATHEMATICS, SHE SPINS BOTH WAYS!





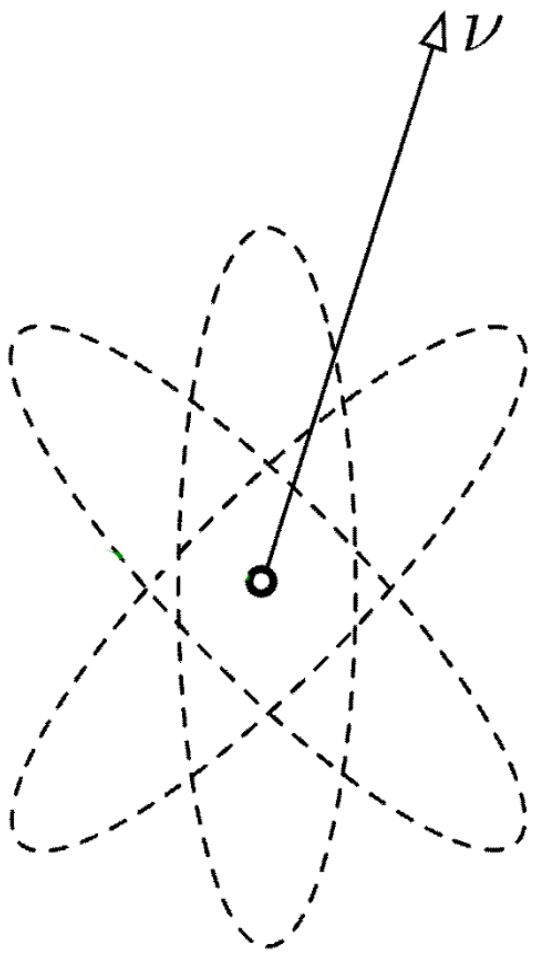
Complete
the phrase…

For every action there is an…

Complete
the phrase…

For every action there is an…
equal and opposite reaction!

Newton's Third
Law of Motion



e^+ positron

e^- electron

ν neutrino

γ quantum/photon (511 keV)

1 NEGATIVE ELECTRON + 1 POSITIVE POSITRON = ZERO!

“SYMMETRY IS WHEN THINGS ARE THE SAME AROUND AN AXIS.”

“SEEING SYMMETRY AND DISCERNING WHEN IT BREAKS, IS A KEY FOR UNDERSTANDING BOTH MATHEMATICS & PHYSICS.”

BIG BANG!

It's as if ŚŪNYA was decompressed,
creating infinite magnitudes
and multitudes from ZERO

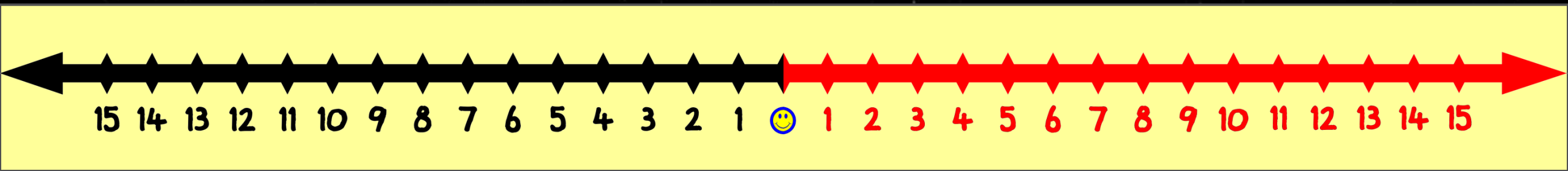
ZERO SUM UNIVERSE CONSERVATION OF MATTER AND ENERGY NEWTON'S THIRD LAW

BRAHMAGUPTA BHĀSKARA SYMMETRY PODOMETIC

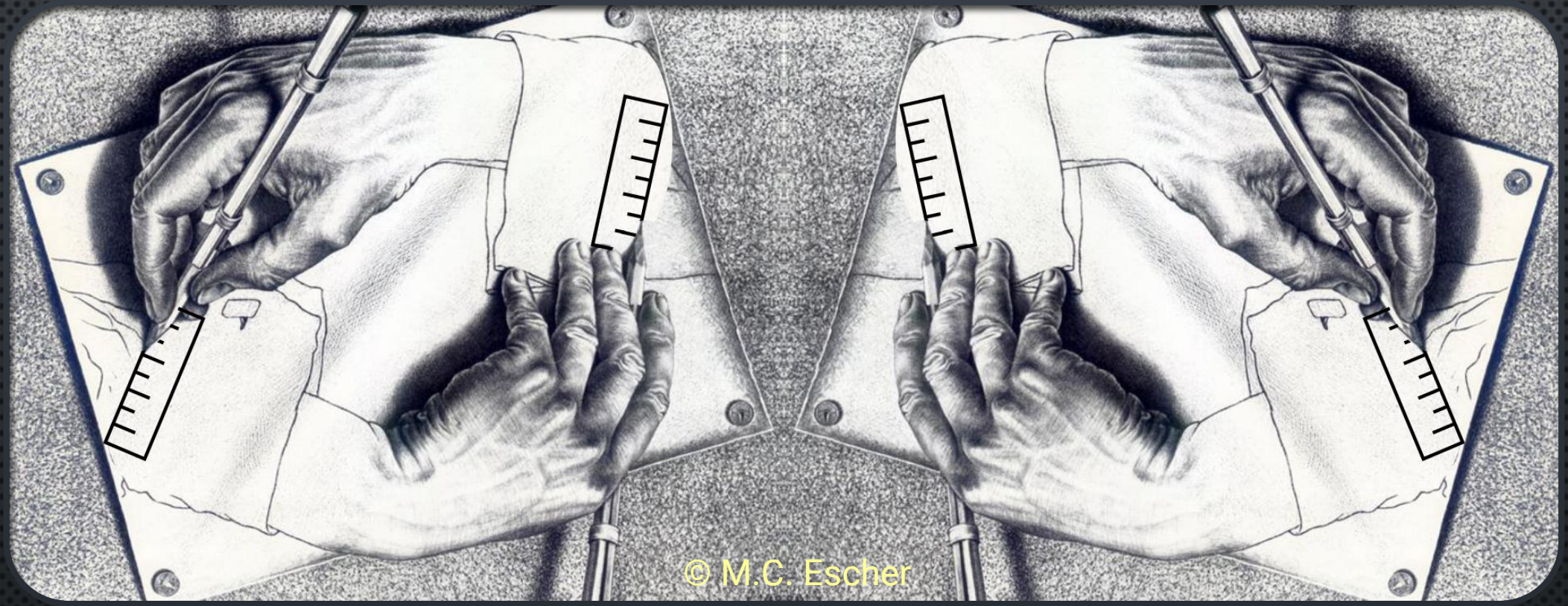
BIG BANG!

ZERO GRAVITY

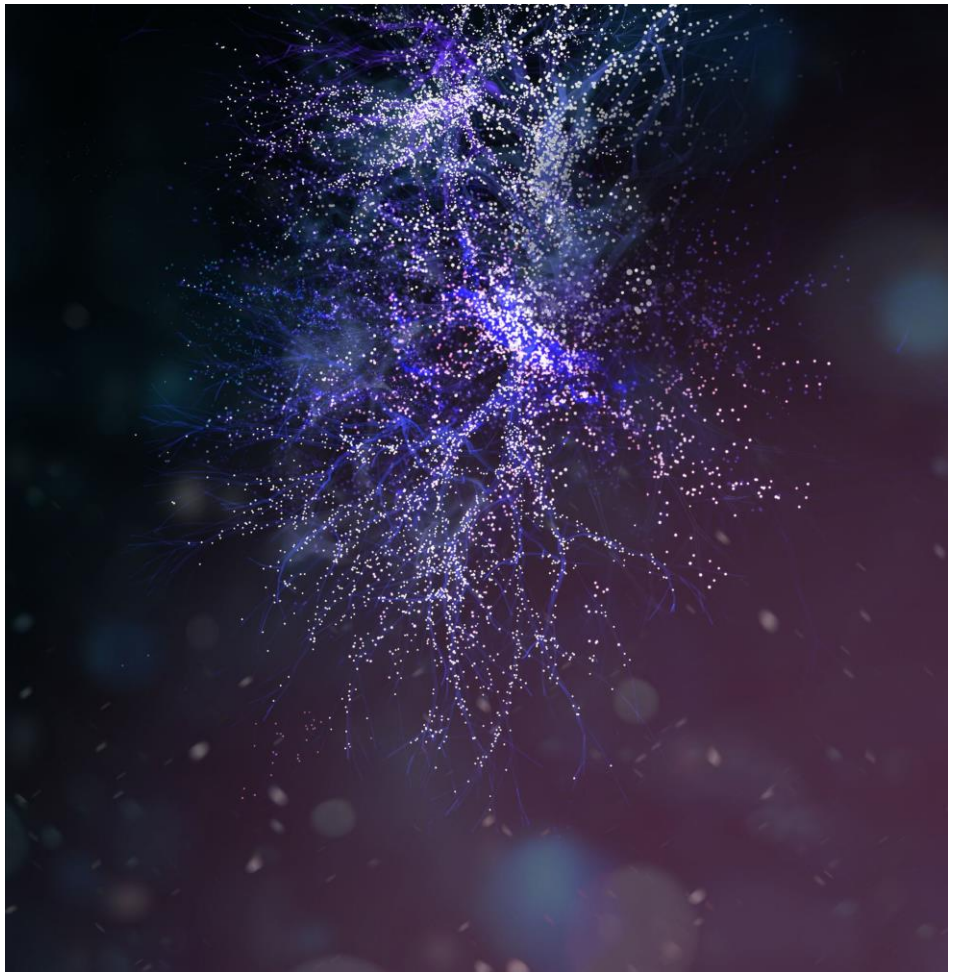
Wherever opposing quantities or forces or directions are equal you will find **ZERO**.



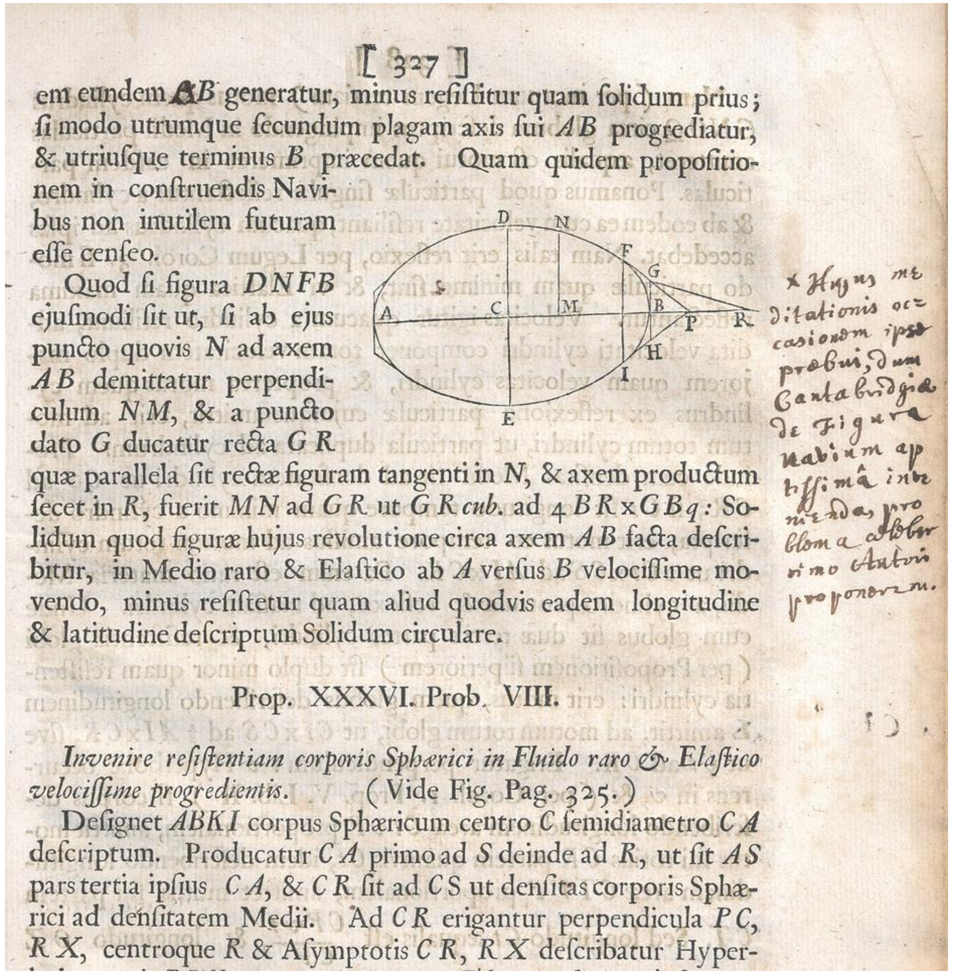
THE ZERO-POINT CHOICE



FROM AN ARBITRARY POINT WHICH DIRECTION DO WE GO?



"PROPORTION IS THE KEY THAT UNLOCKS THE LANGUAGE OF THE UNIVERSE"


[327]

em eundem AB generatur, minus resistitur quam solidum prius; si modo utrumque secundum plagam axis sui *AB* progrediatur, & utriusque terminus *B* præcedat. Quam quidem propositionem in construendis Navibus non inutilem futuram esse censeo.

Quod si figura *DNFB* ejusmodi sit ut, si ab ejus puncto quovis *N* ad axem *AB* demittatur perpendiculum *NM*, & a puncto dato *G* ducatur recta *GR* quæ parallela sit rectæ figuram tangenti in *N*, & axem productum secet in *R*, fuerit *MN* ad *GR* ut *GR cub.* ad $4BR \times GBq$: Solidum quod figuræ hujus revolutione circa axem *AB* facta describitur, in Medio raro & Elastico ab *A* versus *B* velocissime movendo, minus resistetur quam aliud quodvis eadem longitudine & latitudine descriptum Solidum circulare.

x Hujus meditationis occasionem ipse praebui, dum Cantabrigiae de Figura Navium aptissimâ inter me[illegible]ndas, problema celeberrimo Autori proponerem.

Prop. XXXVI. Prob. VIII.

Invenire resistentiam corporis Sphærici in Fluido raro & Elastico velocissime progredientis. (Vide Fig. Pag. 325.)

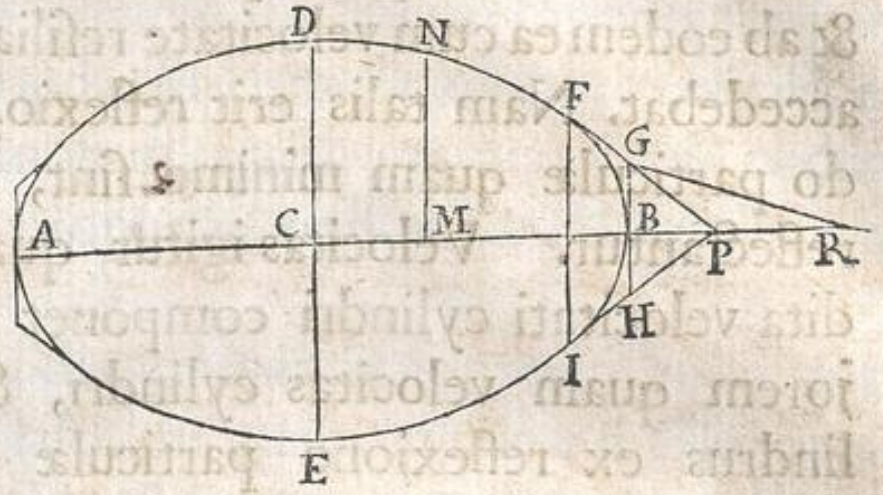
Designet *ABKI* corpus Sphæricum centro *C* semidiametro *CA* descriptum. Producatur *CA* primo ad *S* deinde ad *R*, ut sit *AS* pars tertia ipsius *CA*, & *CR* sit ad *CS* ut densitas corporis Sphærici ad densitatem Medii. Ad *CR* erigantur perpendicula *PC*, *RX*, centroque *R* & Asymptotis *CR*, *RX* describatur Hyper-

WHOSE HANDWRITING?

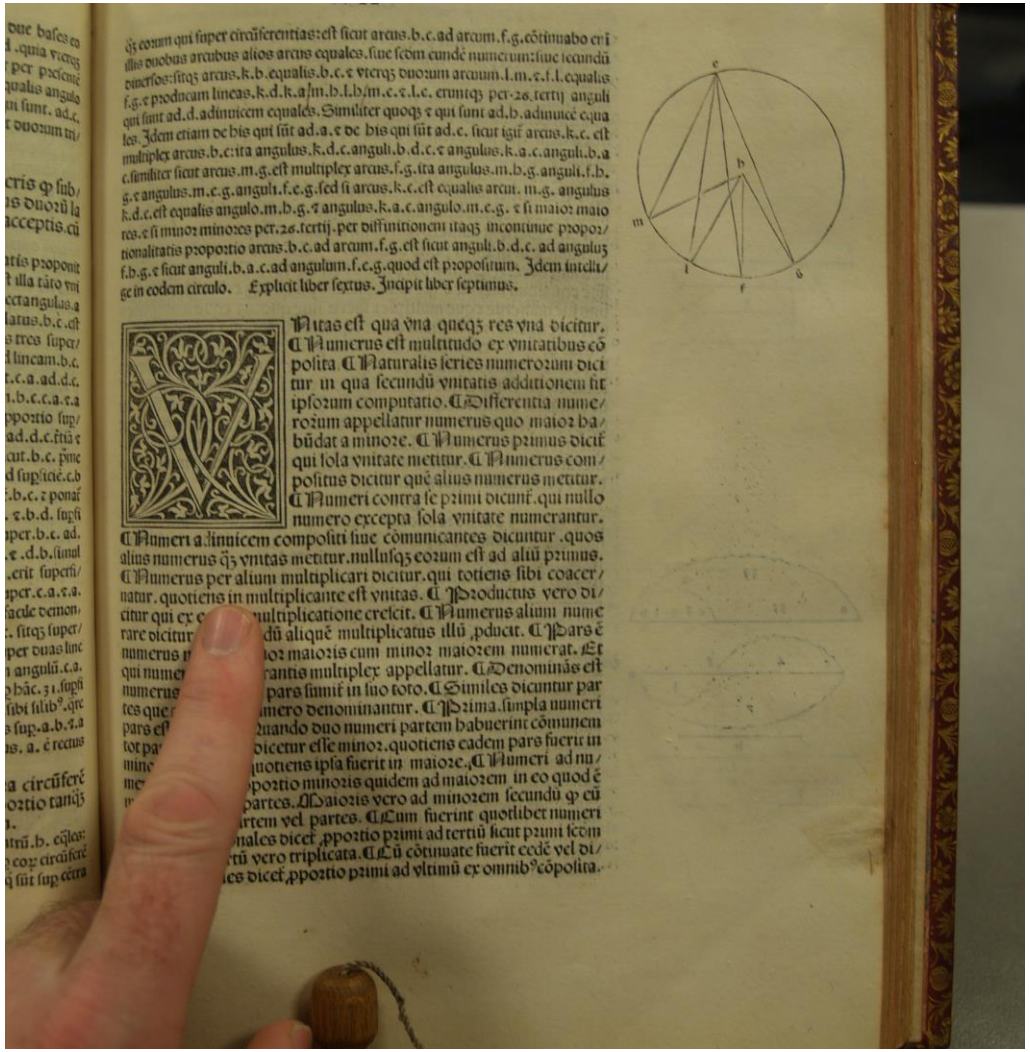
ISAAC NEWTON

em eundem *AB* generatur, minus reſiſtitur quam ſolidum prius; ſi modo utrumque ſecundum plagam axis ſui *AB* progrediatur, & utriuſque terminus *B* præcedat. Quam quidem propoſitionem in conſtruendis Navibus non inutilem futuram eſſe cenſeo.

Quod ſi figura *DNFB* ejuſmodi ſit ut, ſi ab ejus puncto quovis *N* ad axem *AB* demittatur perpendiculum *NM*, & a puncto dato *G* ducatur recta *GR* quæ parallela ſit rectæ figuram tangenti in *N*, & axem productum



* Hujus me ditationis occasionem ipse præbui, dum Cantabrigiæ De Figura Navium ap[illegible]



Vnitas est qua vna quecq; res vna dicitur. ¶ Numerus est multitudo ex vnitatibus cō posita ¶ Naturalis series numerorum dicitur in qua secundū vnitatis additionem fit ipsorum computatio. ¶ Differentia numerorum appellatur numerus quo maior habūdat a minore. ¶ Numerus primus dicit qui sola vnitate metitur. ¶ Numerus compositus dicitur quē alius numerus metitur. ¶ Numeri contra se primi dicunt. qui nullo numero excepta sola vnitate numerantur. ¶ Numeri adinuicem compositi siue cōmunicantes dicuntur .quos alius numerus ōs vnitas metitur. nullusq; eorum est ad aliū primus. ¶ Numerus per alium multiplicari dicitur. qui totiens sibi coacervatur. quotiens in multiplicante est vnitas. ¶ Productus vero dicitur qui ex [illegible] multiplicatione crescit. ¶ Numerus alium numerare dicitur [illegible] dū aliquē multiplicatus illū ,producit. ¶ Pars ē numerus [illegible] maioris cum minor maiorem numerat. Et qui numerat [illegible] rantis multiplex appellatur. ¶ Denominās est numerus [illegible] pars sumit in suo toto. ¶ Similes dicuntur partes que [illegible] mero denominantur. ¶ Prima. simpla numeri pars est [illegible] Quando duo numeri partem habuerint cōmunem tot pa[illegible] dicetur esse minor. quotiens eadem pars fuerit in mino[illegible] quotiens ipsa fuerit in maiore. ¶ Numeri ad numeros [illegible] portio minoris quidem ad maiorem in eo quod ē [illegible] partes. ¶ Maioris vero ad minorem secundū qp cū [illegible] rtem vel partes. ¶ Cum fuerint quotlibet numeri [illegible] nales dicet ,pportio primi ad tertiū sicut primi sctm [illegible] rtiū vero triplicata. ¶ Cū cōmunate fuerit cedē vel di[illegible] es dicet ,pportio primi ad vltimū ex omnib⁹ cōposita.

WHOSE GEOMETRY?

f.b.g.ꝛ sicut anguli.b.a.c.ad angulum.f.c.g.quod est propositum. Idem intelli/
ge in eodem circulo. Explicit liber sextus. Incipit liber septimus.

Vnitas est qua vna queqꝫ res vna dicitur. ¶ Numerus est multitudo ex vnitatibus cō
posita. ¶ Naturalis series numerorum dici
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positus dicitur quē alius numerus metitur.
¶ Numeri contra se primi dicunt. qui nullo
numero excepta sola vnitate numerantur.
¶ Numeri adinuicem compositi siue cōmunicantes dicuntur .quos
alius numerus q̄ꝫ vnitas metitur. nulluſqꝫ eorum est ad aliū primus.
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uatur. quotiens in multiplicante est vnitas. ¶ Productus vero di/
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numerus [illegible]or maioris cum minor maiorem numerat. Et
qui numer[illegible]rantis multiplex appellatur. ¶ Denominās est
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me[illegible]portio minoris quidem ad maiorem in eo quod ē
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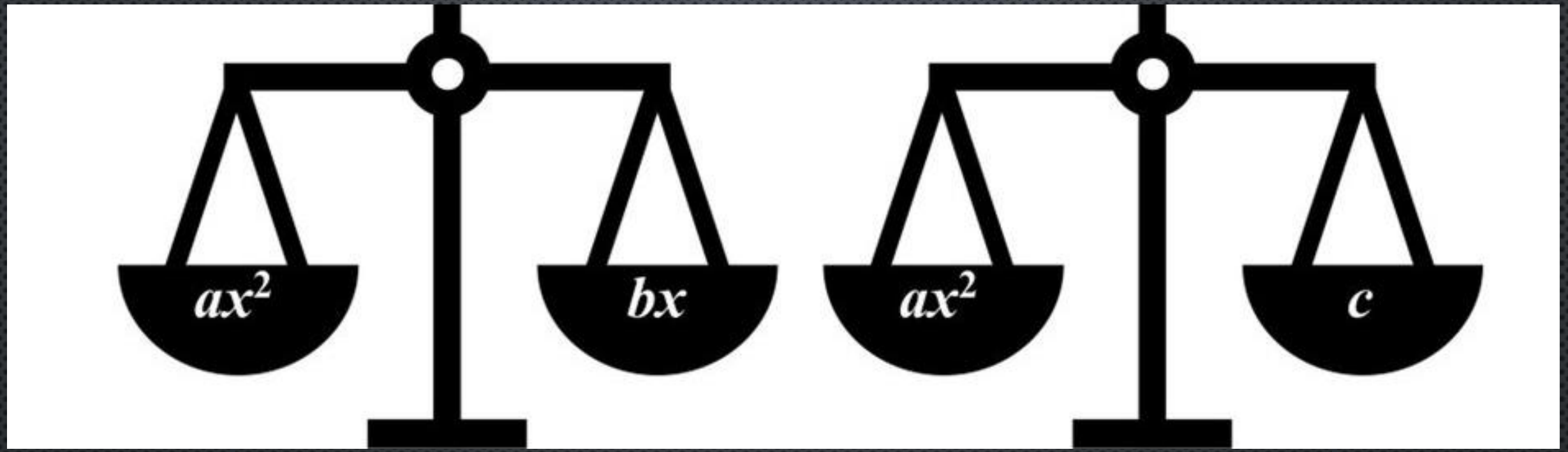
[illegible]rtū vero triplicata. ¶ Cū cōtinuate fuerit eedē vel di/
[illegible]es dicet ,pportio primi ad vltimū ex omnib⁹ cōposita.

EUCLID

Arabic algebra entailed steps to arrive at co-equal polynomials. The following is an anachronistic styling as modern equations.

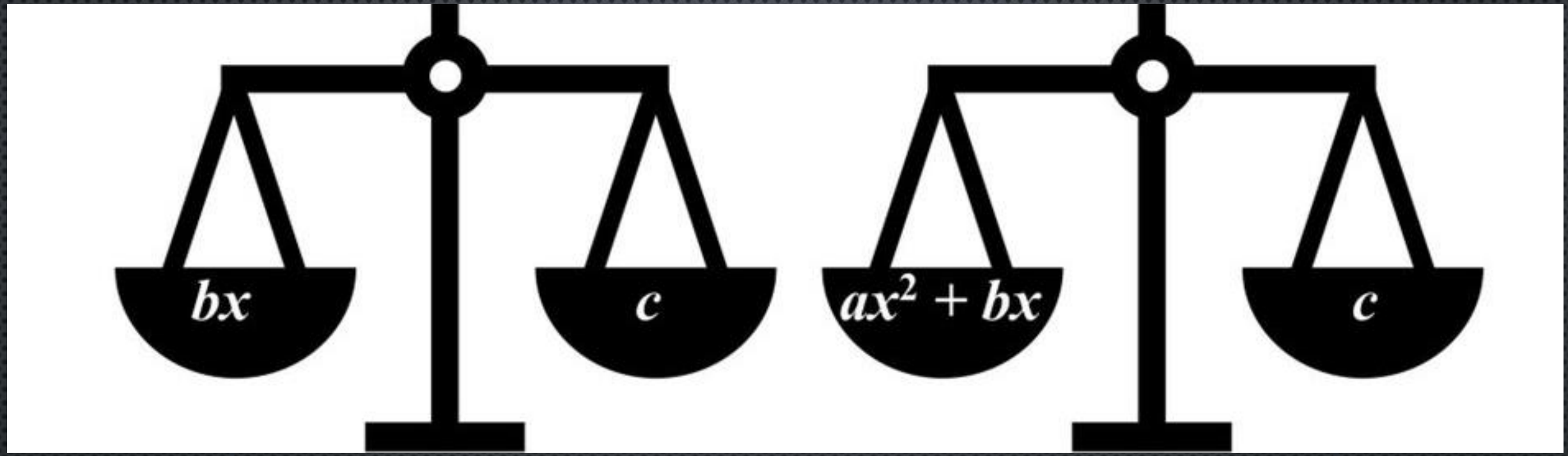
$$ax^2 = bx \qquad ax^2 = c$$

AL-KHWARIZMI'S 6 EQUATION TYPES



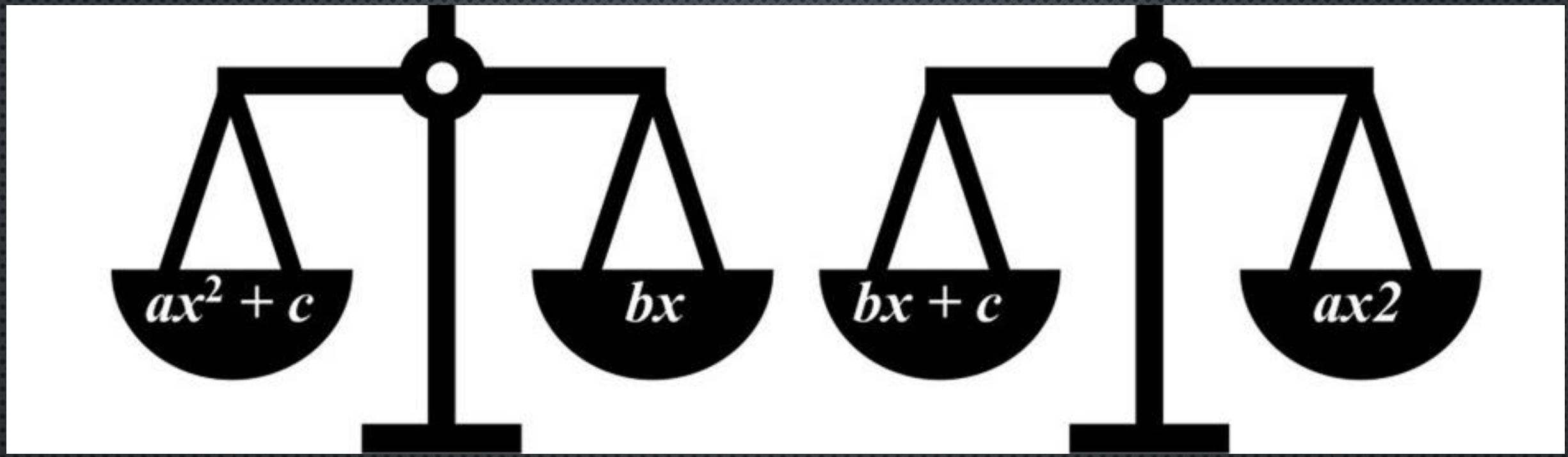
$$ax^2 = bx \mid ax^2 = c$$

AL-KHWARIZMI'S 6 EQUATION TYPES



$$bx = c \mid ax^2 + bx = c$$

AL-KHWARIZMI'S 6 EQUATION TYPES

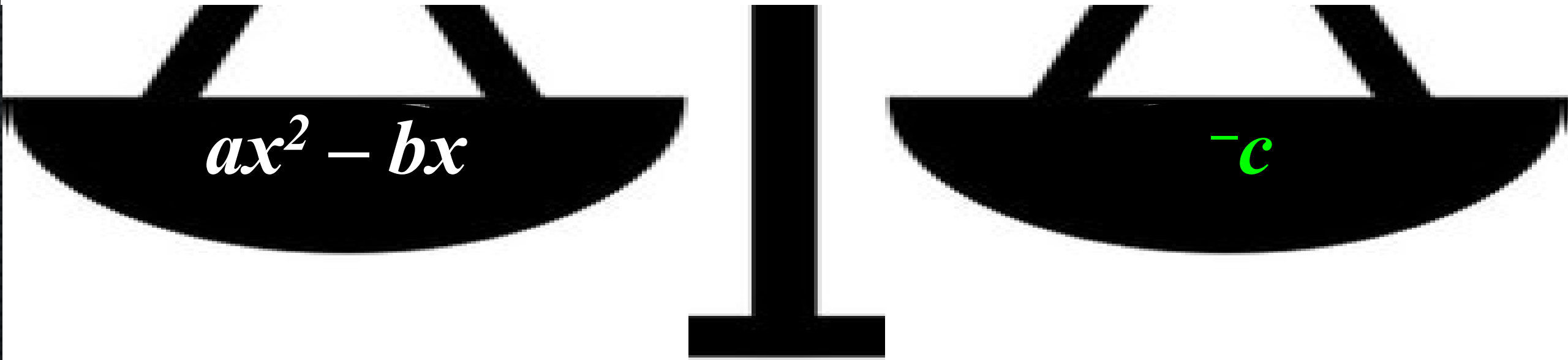


$$ax^2 + c = bx \quad | \quad bx + c = ax^2$$

AL-KHWARIZMI'S 6 EQUATION TYPES

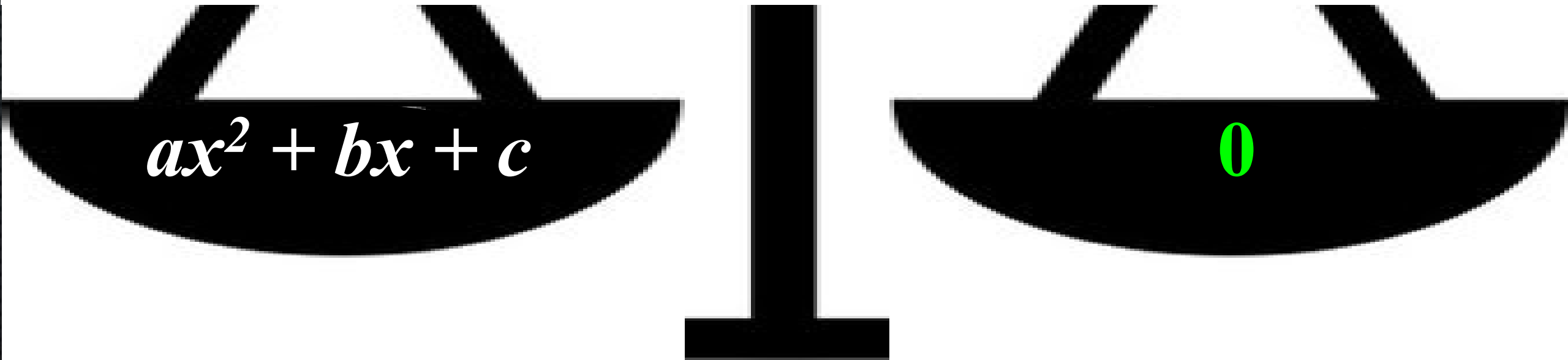
$ax^2 - bx = \bar{}c$

MISSING!



$ax^2 + bx + c = 0$

MISSING!



Negative terms did not occur in Arabic algebra, yet you wouldn't know that today.

AL-KHWĀRIZMĪ'S ALGEBRA TEXT C. 820 CE

لكتاب المختصر في حساب الجبر والمقابلة

Al-Kitāb al-mukhtasar fī hisāb **al-jabr** wa'l-muqābala

The Compendious Book on Calculation by Completion [or Restoration] and Balancing.

In mathematical language, the verb [**jabr**] means… … to **transpose NEGATIVE quantities** to the opposite side by changing their signs. The **NEGATIVE quantity thus removed**…

KHWĀRAZMI, A. A. A. M. A. M., & ROSEN, F. A. (1831). THE ALGEBRA OF MOHAMMED BEN MUSA. LONDON: PRINTED FOR THE ORIENTAL TRANSLATION FUND.

The usual **meaning of jabr** in mathematical treatises is: adding equal terms to both sides of an equation in order to **eliminate NEGATIVE terms**.

WAERDEN, B. L. (2013). A HISTORY OF ALGEBRA: FROM AL-KHWĀRIZMĪ TO EMMY NOETHER. BERLIN: SPRINGER BERLIN.

Al-jabr means “restoration” or “completion”, that is, **removing NEGATIVE terms**, by transposing them to the other side of the equation to make them positive

DEVLIN, K. (2012). *THE MAN OF NUMBERS: FIBONACCI'S ARITHMETIC REVOLUTION*. LONDON: BLOOMSBURY.

Negative terms equidistant from **zero** as opposite **positive** terms did **NOT occur in Arabic algebra**.

Brahmagupta's algebraic definition of **zero** as the **sum** of **equal and opposite quantities** is **absent**.

Brahmagupta's 5 Addition Sutras

AS1 positive plus positive is positive

AS2 negative plus negative is negative

AS3 positive plus negative is the difference between the positive and negative

AS4 when positive and negative are equal the sum is zero

AS5
positive plus zero is positive
negative plus zero is negative
zero plus zero is zero

Brahmagupta's 5 Addition Sutras

AS1 positive plus positive is positive

AS2

AS3

AS4

AS5

ZEROS & NEGATIVE TERMS ARE IN THESE SUTRAS!

Brahmagupta's 5 Addition Sutras

धनयोर्धनम्ऋणमृणयोः धनर्णयोरन्तरं समैक्यं खम् ऋणमैक्यं च धनमृणधनशून्ययोः शून्ययोः शून्यम्

AS1 positive plus positive is positive

AS2 negative plus negative is negative

AS3 positive plus negative is the difference between the positive and negative

AS4 when positive and negative are equal the sum is zero

AS5
positive plus zero is positive
negative plus zero is negative
zero plus zero is zero

Brahmagupta's 5 Subtraction Sutras

ऊनमधिकाद्विशोध्यं धनं धनाद्ऋणमृणाद्अधिकमूनात् व्यस्तं तदन्तरं स्याद्दृणं धनं धनमृणं भवति
शून्यविहीनमृणमृणं धनं धनं भवति शून्यमाकाशम् शोध्यं यदा धनमृणाद्ऋणं धनाद्वा तदा क्षेप्यम

SS1 A smaller positive subtracted from a larger positive is positive.

SS2 A smaller negative subtracted from a larger negative is negative.

SS3 If a larger negative or positive is to be subtracted from a smaller negative or positive, the sign of their difference is reversed – negative becomes positive and positive negative.

SS4
A negative minus zero is negative,
a positive minus zero is positive,
zero minus zero is zero.

SS5 When a positive is to be subtracted from a negative or a negative from a positive, then it is to be added.

Brahmagupta's 4 Multiplication Sutras

ऋणमृणधनयोर्घातो धनमृणयोः धनवधो धनं भवति
शून्यर्णयोः खधनयोः खशून्ययोर्वा वधः शून्यम्

MS1 The product of a negative and a positive is negative.

MS2 The product of two negatives is positive.

MS3 The product of two positives is positive.

MS4
The product of zero and a negative,
of zero and a positive, or
of two zeros is zero.

Brahmagupta's 4 Division Sutras

धनभक्तं धनम् ऋणहृतमृणं धनं भवति खं खभक्तं खम्
भक्तमृणेन धनमृणं धनेन हृतम् ऋणमृणं भवति
खोद्धृतमृणं धनं वा तच्छेदं खमृणधनविभक्तं वा
ऋणधनयोर्वर्गः स्वं खं खस्य पदं कृतिर्यत् तत्

DS1 A positive divided by a positive is positive.

DS2 A negative divided by a negative is positive.

DS3 A positive divided by a negative is negative.

DS4 A negative divided by a positive is negative.

Acknowledgement: I am grateful to Avinash Sathaye, K. Ramasubramanian, Clemency Montelle, Kim Plofker and Agathe Keller. Analysis, interpretation *(& any mistakes)* by Jonathan J. Crabtree.

Brahmagupta's 5 Addition Sutras

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AS1 positive plus positive is positive

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AS3 positive plus negative is the difference between the positive and negative

AS4 when **positive** and **negative** are equal the sum is **zero**

AS5 positive plus zero is positive
negative plus zero is negative
zero plus zero is zero

Brahmagupta's 5 Addition Sutras

धनयोर्धनम्ऋणमृणयोः धनर्णयोरन्तरं समैक्यं खम् ऋणमैक्यं च धनमृणधनशून्ययोः शून्ययोः शून्यम्

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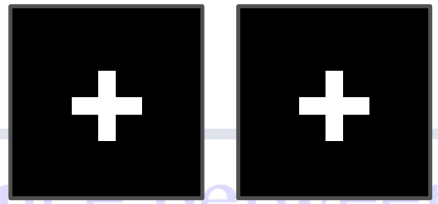
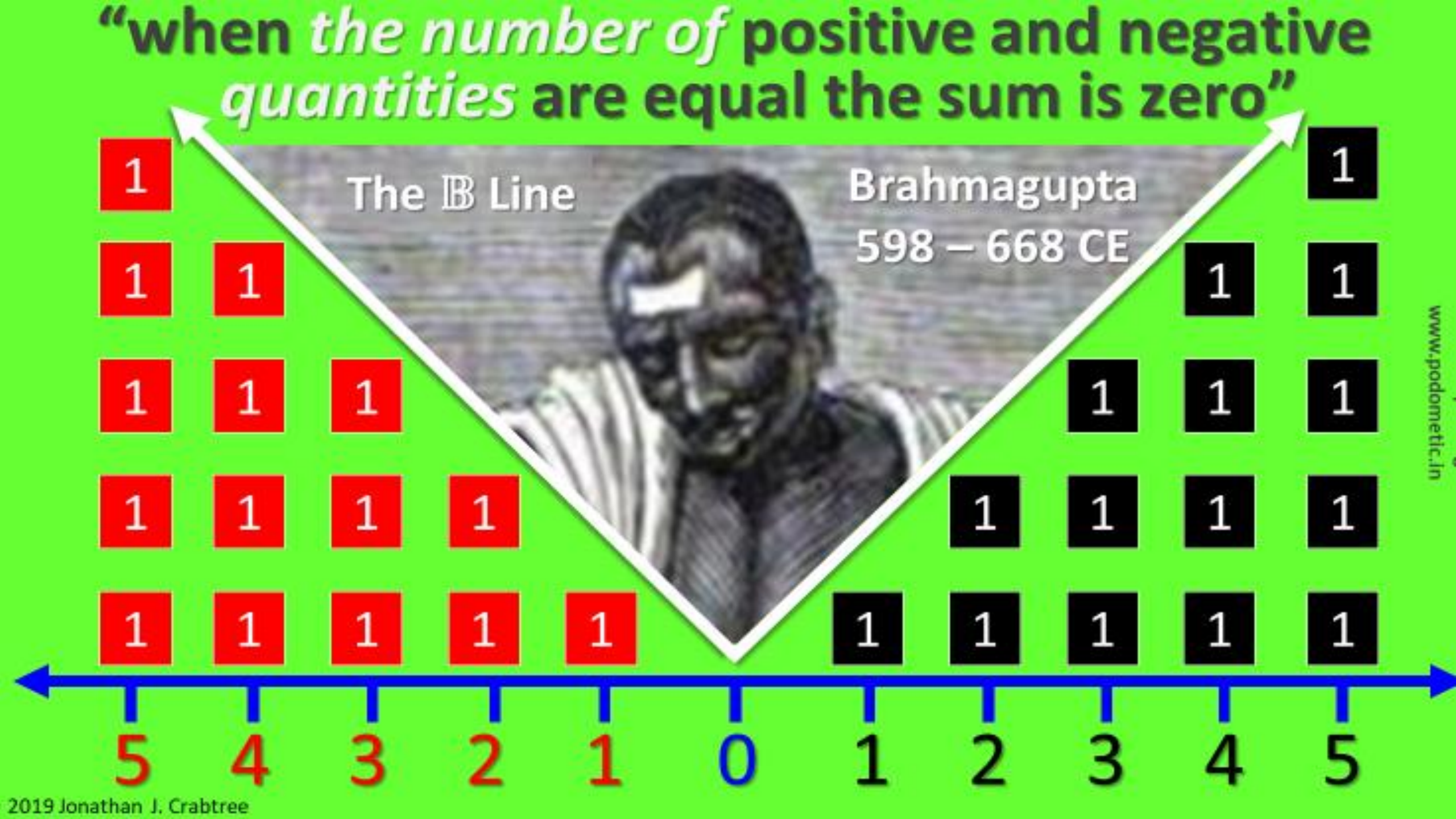


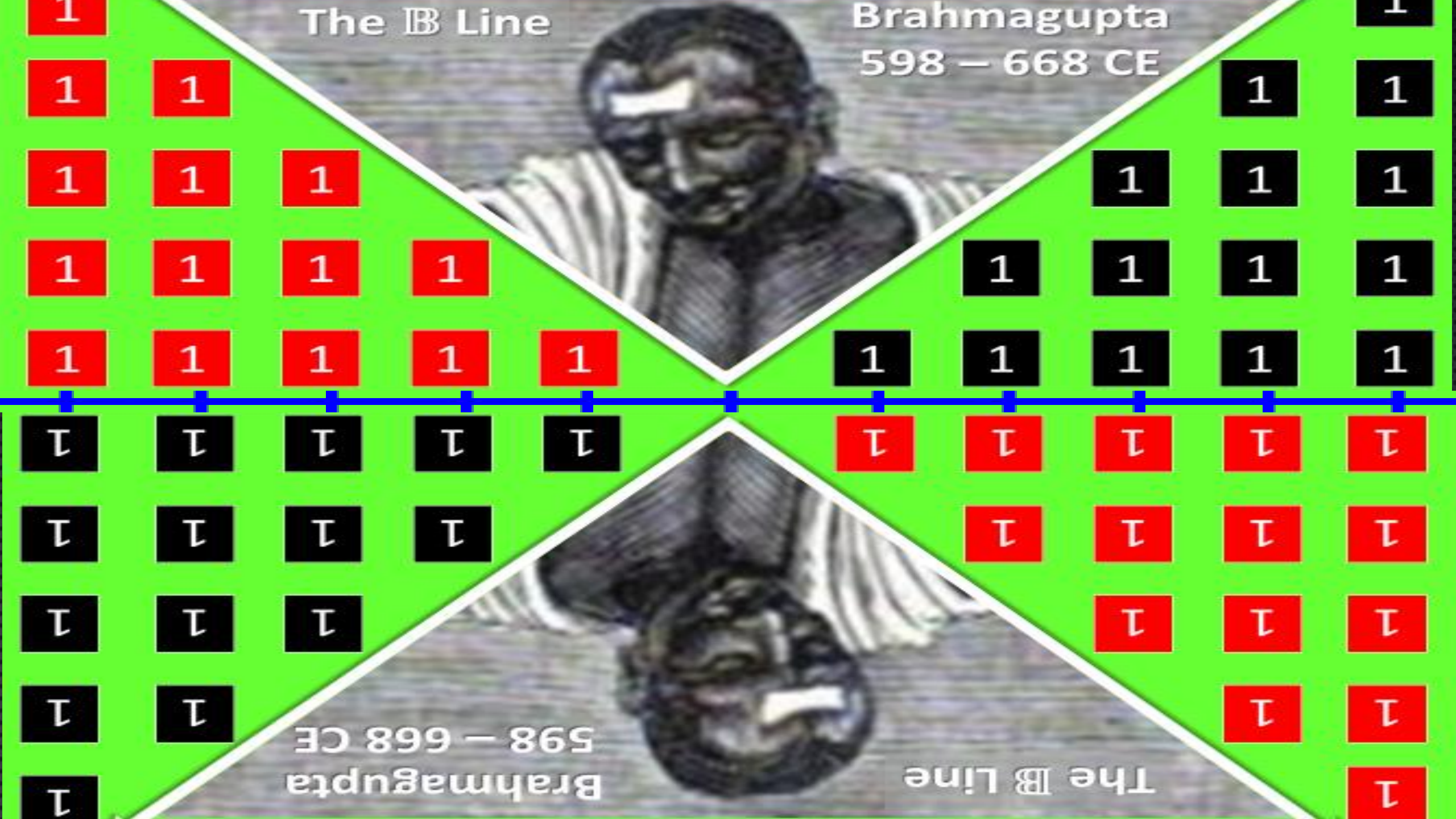
Add Integers to Zero

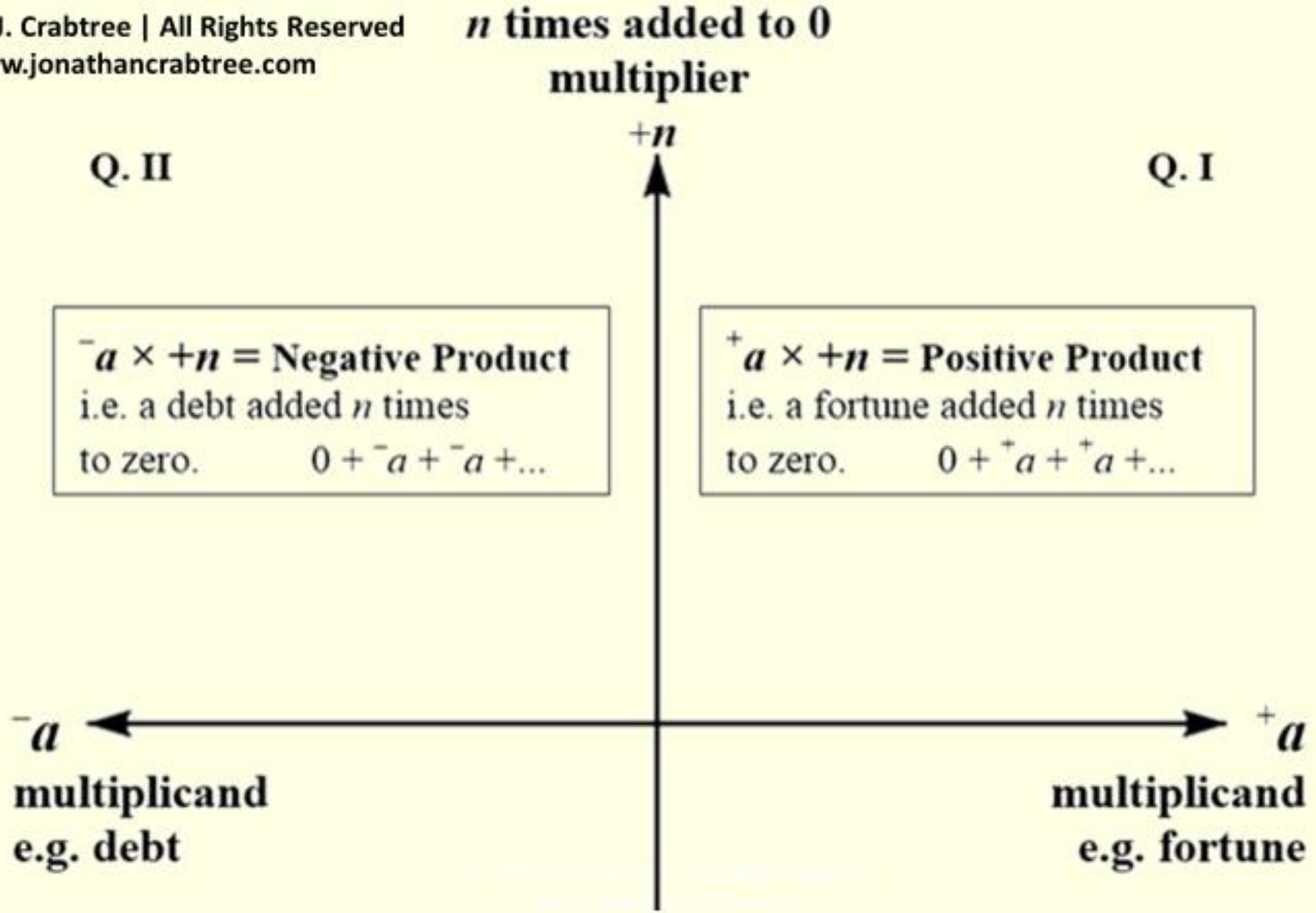
"when *the number of* positive and negative *quantities* are equal the sum is zero"

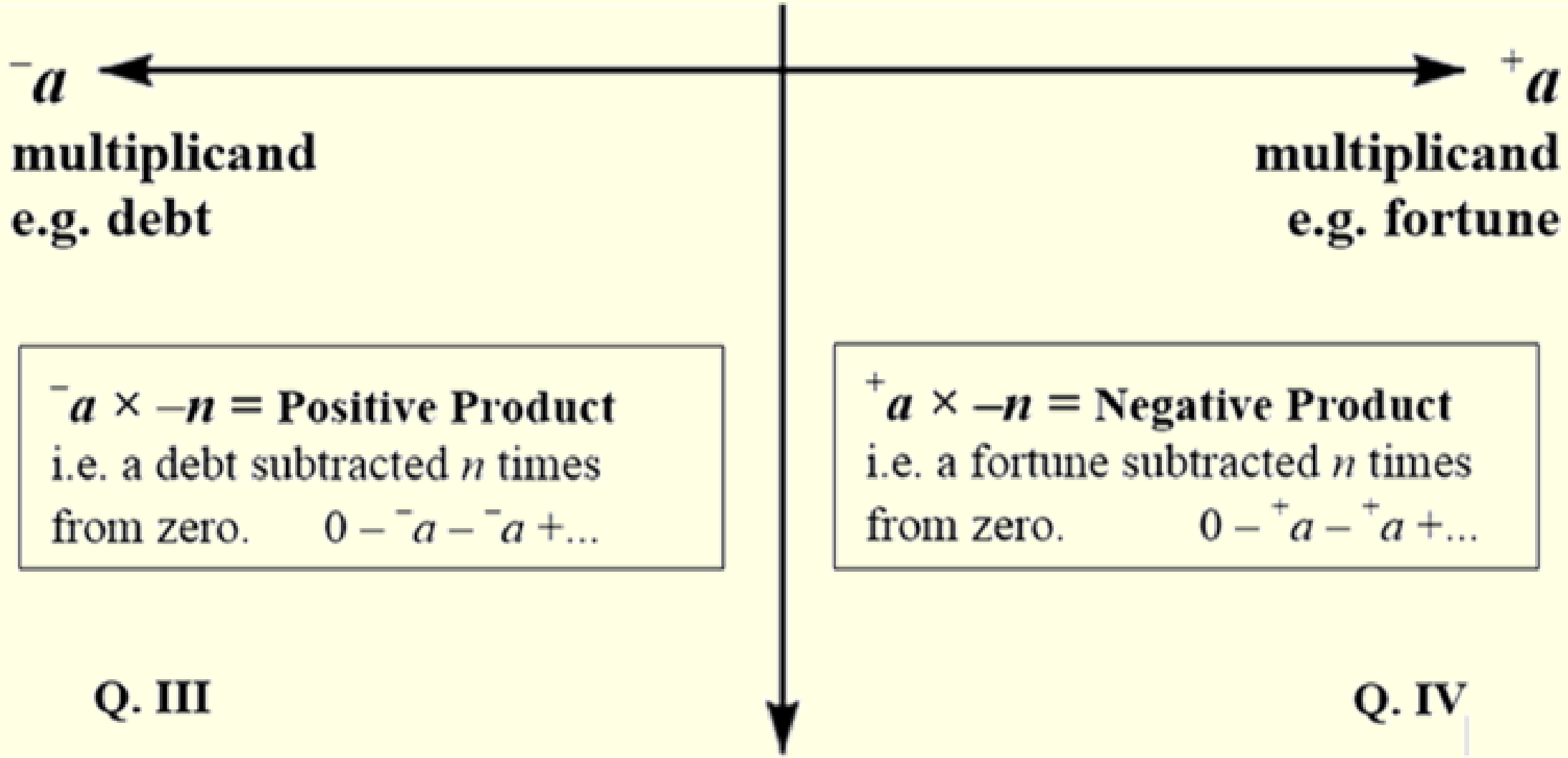


Add Integers to Zero

Subtract Integers from Zero


The 𝔹 Line
Brahmagupta
598 – 668 CE





Q. I

$^{+}a \times +n =$ **Positive Product**

i.e. a fortune added n times to zero. $0 + {}^{+}a + {}^{+}a + \ldots$

Q. II

$^{-}a \times +n$ = **Negative Product**

i.e. a debt added n times

to zero. $0 + {}^{-}a + {}^{-}a + \ldots$

$\bar{a} \times -n \equiv$ **Positive Product**

i.e. a debt subtracted n times

from zero. $0 - \bar{a} - \bar{a} + \ldots$

Q. III

$-n$

$^{+}a \times -n$ = **Negative Product**

i.e. a fortune subtracted n times from zero. $0 - {}^{+}a - {}^{+}a + \ldots$

Q. IV

$-n$

Addition of Integers to Zero

Negatives Added N Times to Zero									Multiplier	Positives Added N Times to Zero								
81	72	63	54	45	36	27	18	9	+9	9	18	27	36	45	54	63	72	81
72	64	56	48	40	32	24	16	8	+8	8	16	24	32	40	48	56	64	72
63	56	49	42	35	28	21	14	7	+7	7	14	21	28	35	42	49	56	63
54	48	42	36	30	24	18	12	6	+6	6	12	18	24	30	36	42	48	54
45	40	35	30	25	20	15	10	5	+5	5	10	15	20	25	30	35	40	45
36	32	28	24	20	16	12	8	4	+4	4	8	12	16	20	24	28	32	36
27	24	21	18	15	12	9	6	3	+3	3	6	9	12	15	18	21	24	27
18	16	14	12	10	8	6	4	2	+2	2	4	6	8	10	12	14	16	18
9	8	7	6	5	4	3	2	1	+1	1	2	3	4	5	6	7	8	9
$^{-}9$	$^{-}8$	$^{-}7$	$^{-}6$	$^{-}5$	$^{-}4$	$^{-}3$	$^{-}2$	$^{-}1$	0	$^{+}1$	$^{+}2$	$^{+}3$	$^{+}4$	$^{+}5$	$^{+}6$	$^{+}7$	$^{+}8$	$^{+}9$
9	8	7	6	5	4	3	2	1	-1	1	2	3	4	5	6	7	8	9
18	16	14	12	10	8	6	4	2	-2	2	4	6	8	10	12	14	16	18
27	24	21	18	15	12	9	6	3	-3	3	6	9	12	15	18	21	24	27
36	32	28	24	20	16	12	8	4	-4	4	8	12	16	20	24	28	32	36
45	40	35	30	25	20	15	10	5	-5	5	10	15	20	25	30	35	40	45
54	48	42	36	30	24	18	12	6	-6	6	12	18	24	30	36	42	48	54
63	56	49	42	35	28	21	14	7	-7	7	14	21	28	35	42	49	56	63
72	64	56	48	40	32	24	16	8	-8	8	16	24	32	40	48	56	64	72
81	72	63	54	45	36	27	18	9	-9	9	18	27	36	45	54	63	72	81
Negatives Subtracted N Times from Zero									Multiplier	Positives Subtracted N Times from Zero								

Negative

Positive

Positive

Negative

SIDE OF NEGATIVE MULTIPLICANDS

SIDE OF POSITIVE MULTIPLICANDS

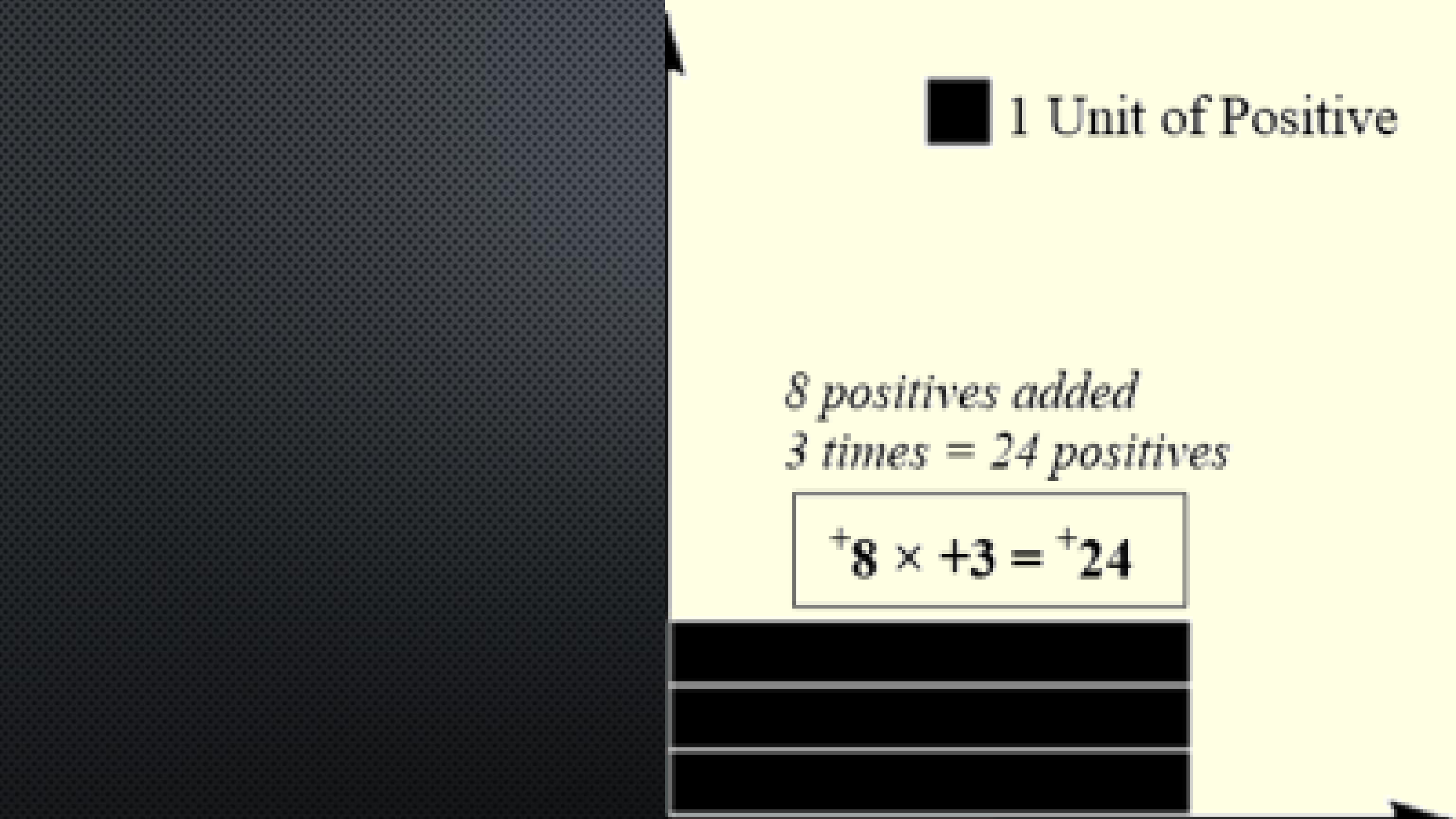
Subtraction of Integers from Zero

Representations of Negative and Positive Quantities on a 'Brahmaguptan Plane' for India's Primary Classes

Jonathan J Crabtree

www.j.mp/BrahmaguptanPlane

Abstract: *Children's fear of maths is often associated with the introduction of negative numbers. By way of example, asking adult non-mathematicians for the answer to 'negative seven minus negative four' usually results in a wrong answer. However, asking the same question to 12-year-old children in the form What does seven negatives minus four negatives equal? usually results in the right answer. Why is the difference in comprehension so dramatic? In the problematic expression negative seven minus negative four the syntactic structure is adjective adjective verb adjective adjective. With the absence of a noun, the meaning of such maths for most children is lost. Instead, children (and adults) cling to rules memorised without meaning, such as 'two minuses make a plus'. So, what can we do? The answer is simple. We return to 7th Century writings of India, where we discover the astronomer Brahmagupta documented 'adjective-noun' style laws of sign, not for abstract numbers, but for positive quantities, negative quantities and zero. With this insight, we depict simple object-oriented representations of integer arithmetic involving positive and negative quantities. Such a quantitative pedagogy is concrete in nature, yet isomorphic to 'signed numbers.' Therefore, a solid intuitive foundation of integer arithmetic can be laid. Upon this foundation more abstract structures can be built. The integer teaching model that emerges is called the 'Brahmaguptan Plane'.*


1 Unit of Positive
8 positives added
3 times = 24 positives
+8 × +3 = +24

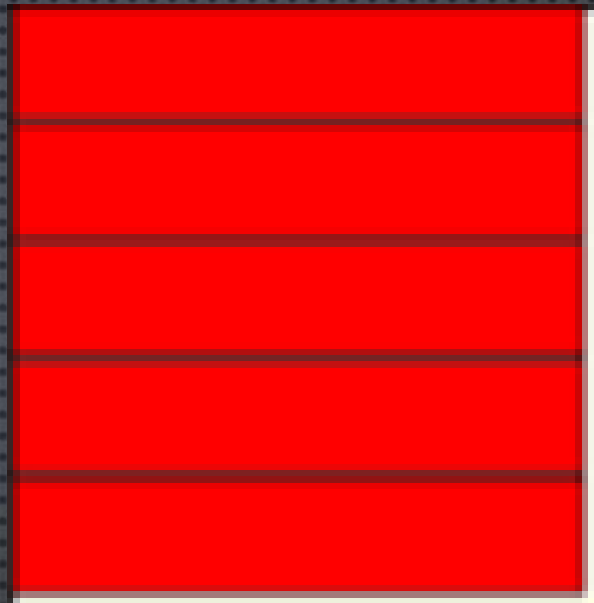
1 Unit of Negative

10 negatives added
5 times = 50 negatives

$$^{-}10 \times +5 = {}^{-}50$$

$$^{-}3 \times -6 = {}^{+}18$$

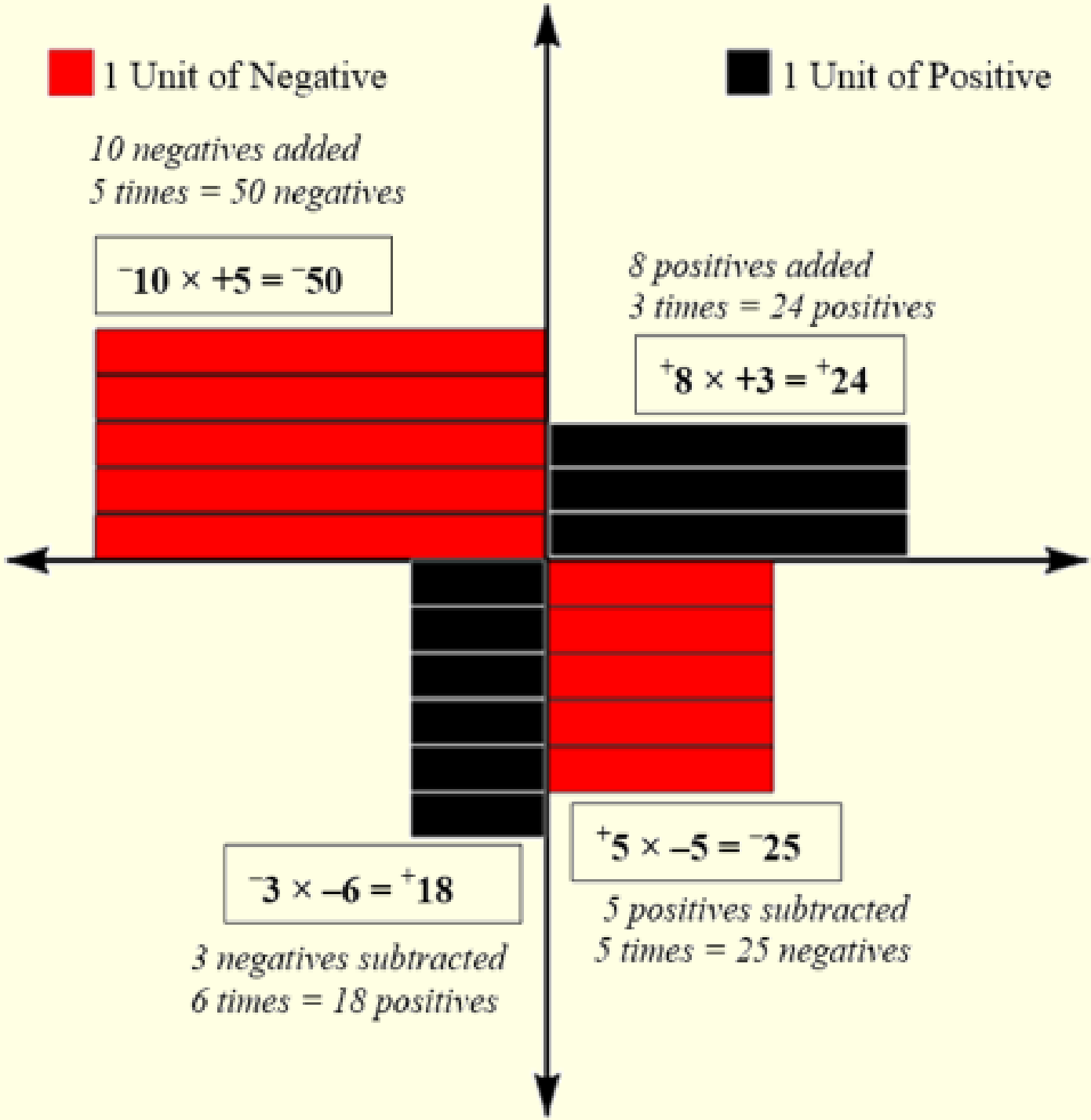
3 negatives subtracted
6 times = 18 positives



$$^{+}5 \times -5 = {}^{-}25$$

5 positives subtracted
5 times = 25 negatives

The Brahmaguptan Plane with both positive and negative areas.



Brahmagupta's 5 Subtraction Laws

SL1 A smaller **positive** subtracted from a larger positi… …sitive.

SL2 A smaller **negative** subtracted from a larger negativ… …tive.

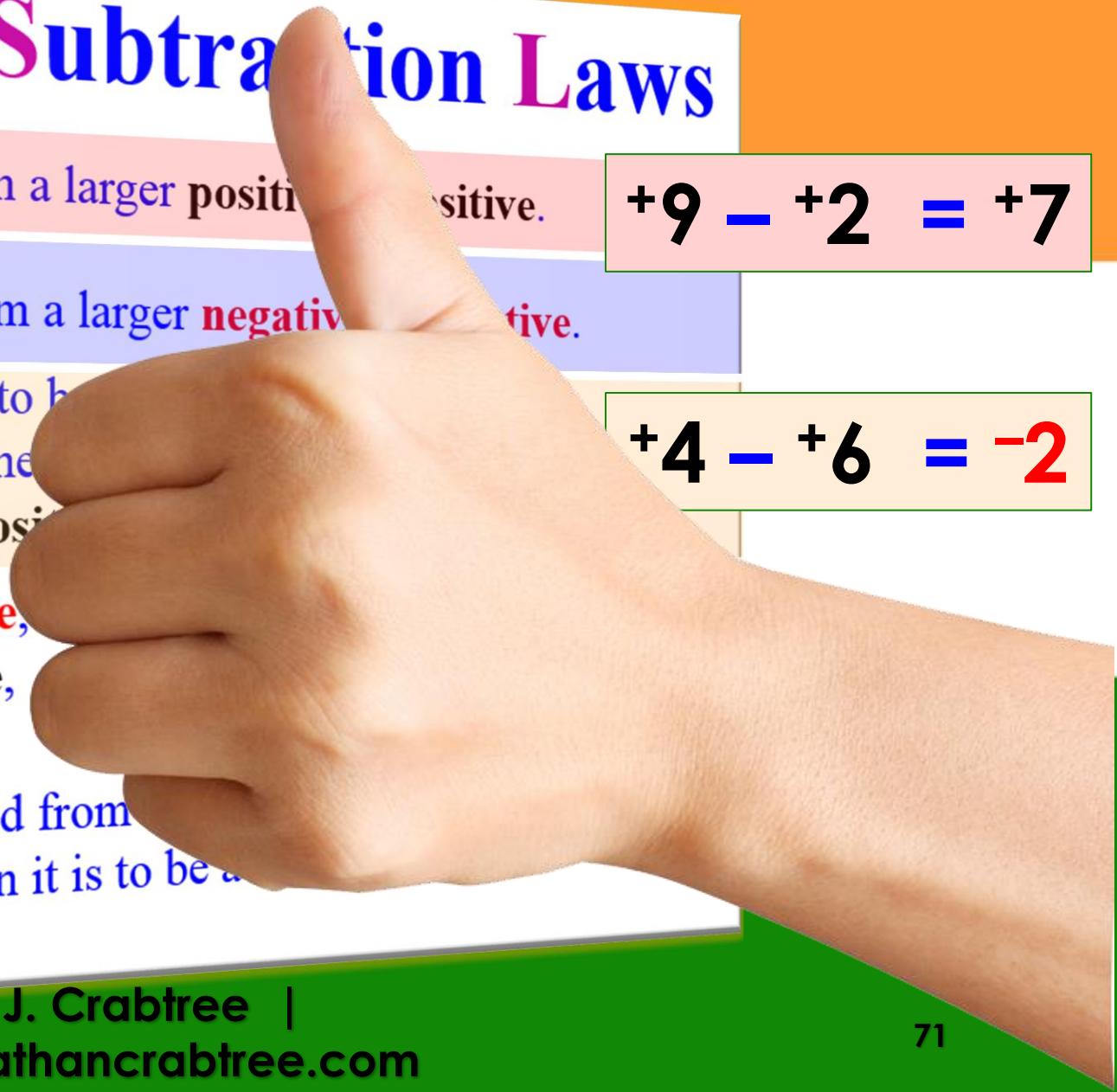
SL3 If a larger **negative** or **positive** is to b…
negative or **positive**, the sign of the…
negative becomes **positive** and posi…

SL4 A **negative** minus **zero** is **negative**,
a **positive** minus **zero** is **positive**,
zero minus **zero** is **zero**.

SL5 When a **positive** is to be subtracted from…
or a **negative** from a **positive**, then it is to be …

$^{+}9 - {}^{+}2 = {}^{+}7$

$^{+}4 - {}^{+}6 = {}^{-}2$




Brahmagu
SL1
A smaller
SL2
A small
SL3
If a lar
negati
negati
SL4
A negat
a positiv
zero
SL5
When a posi
or a negative fi
n Laws
e.
e.
Smaller Negative!
$^{-}8 - {}^{-}5 = {}^{-}3$
$^{-}3 - {}^{-}7 = {}^{+}4$
Larger Negative!
72
www.jonathancrabtree.com

Al-Khwārizmī (c. 780-850)

I had seen that the Indians had set up **9 symbols** in their universal system of numbering…

Al-Khwārizmī did not mention zero

Al-Khwārizmī (c. 780-850)

So they made **9 symbols**, which are these:
9 8 7 6 5 4 3 2 1.

And … **every number is put together above one**.

"Algorizmi said: since I had seen that the Indians had set up IX symbols..." Crossley, John N, and Henry, Alan S. (1990) *Thus Spake Al-Kẖwārizmī: A Translation of the Text of Cambridge University Library Ms. Ii. Vi. 5*. Historia Mathematica. P. 110-111

Al-Khwārizmī (c. 780-850)

… **one** is the root of all number and **is outside number**.

It is the root of number because every number is found by it.

But it [one] is outside number because it is found by itself, I mean, without any other number.

Al-Khwārizmī did not mention zero and did not consider one a number.

Al-Uqlidisi (c. 920-980)

Why are the Hindi letters nine,

So much for the nine letters

zero the aim is only to occupy the place

We multiply the letter… to occupy the place,

tell that there is a place and that it is empty.

Saidan, Ahmad S. (1978) *The Arithmetic of Al-Uqlídisí: The Story of Hindu-Arabic Arithmetic As Told in Kitab Al-Fusul Fi Al-Hisab Al-Hindi.* Reidel, Dordrecht. P. 186

Ibn al-Bannā (1256 – 1321)

"Then you **add** each digit of one of the numbers to the corresponding digit of the other. If there is **nothing** in the corresponding place, then the answer is the number, **as if it had a corresponding number.**"

Courtesy Jeff Oaks via email

Ibn al-Bannā (1256 – 1321)

"Then you **add** each digit of one of the numbers to the corresponding digit of the other. If there is **nothing** in the corresponding place, then the answer is the number, **as if it had a corresponding number.**"

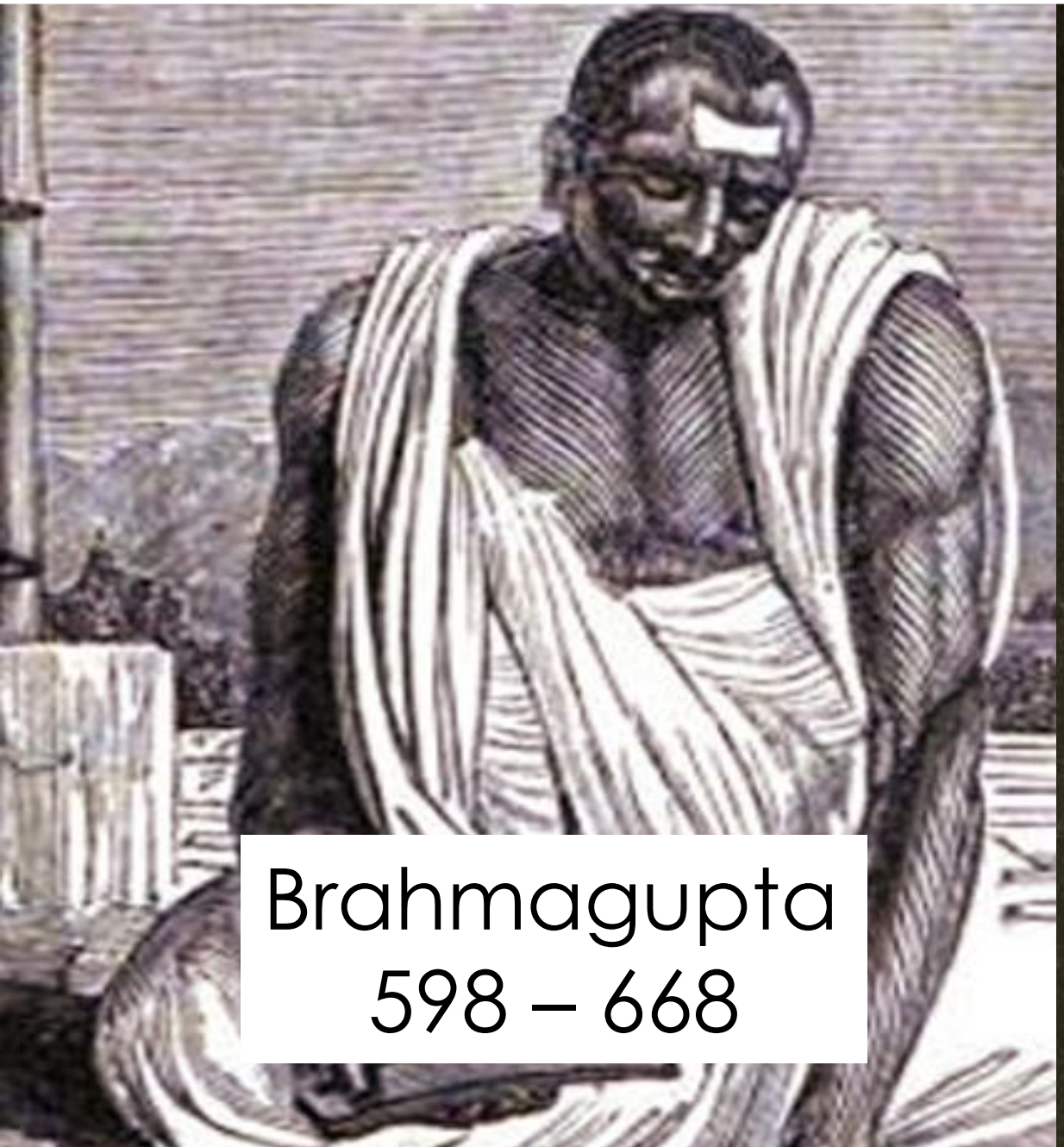
i.e. no addition operation occurs with zero

Al-Hawārī (c. 1305)

"**multiplying** the number **by** the **zero** or the zero by the number is identical. It comes from **voiding the number** or **duplicating zero**. **Neither of these gives a number**".

Courtesy Jeff Oaks via email

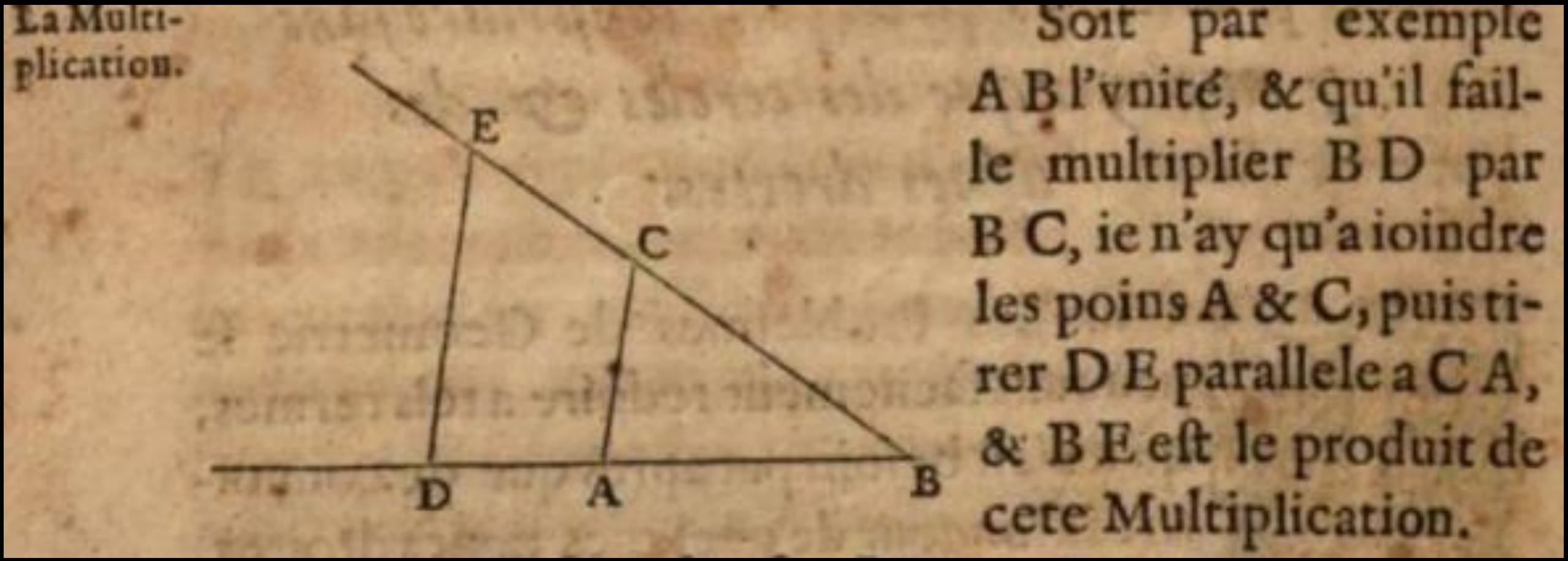
Brahmagupta's ideas were not applied 1000 years later, yet should have been.



Brahmagupta
598 – 668

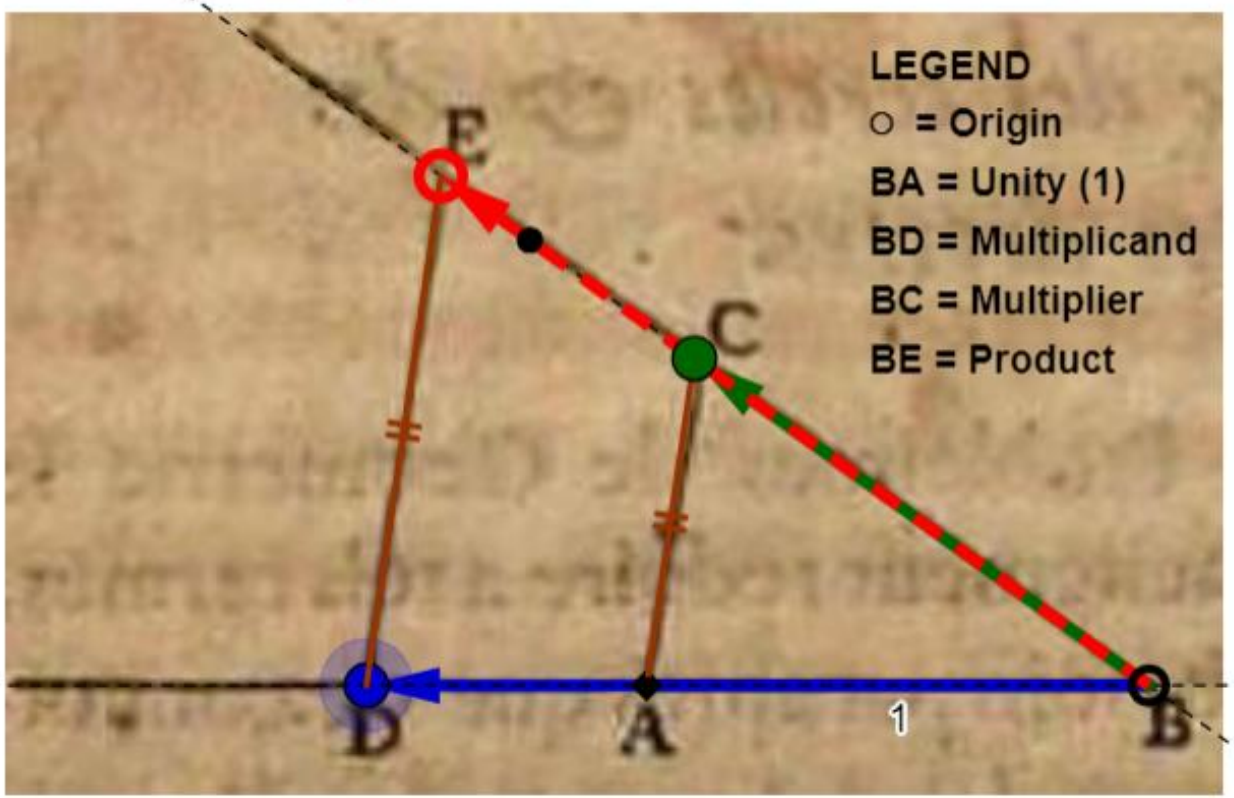
René Descartes
1596 – 1650

Applying Indian Logic to Descartes's Multiplication



"For example, let AB be taken as unity, (1), and let it be required to multiply BD (the multiplicand) by BC (the multiplier), I have only to join the points A and C, and draw DE parallel to AC; and BE is the product of this Multiplication."

Indian Logic Meets Descartes' 1637 Multiplication Model.



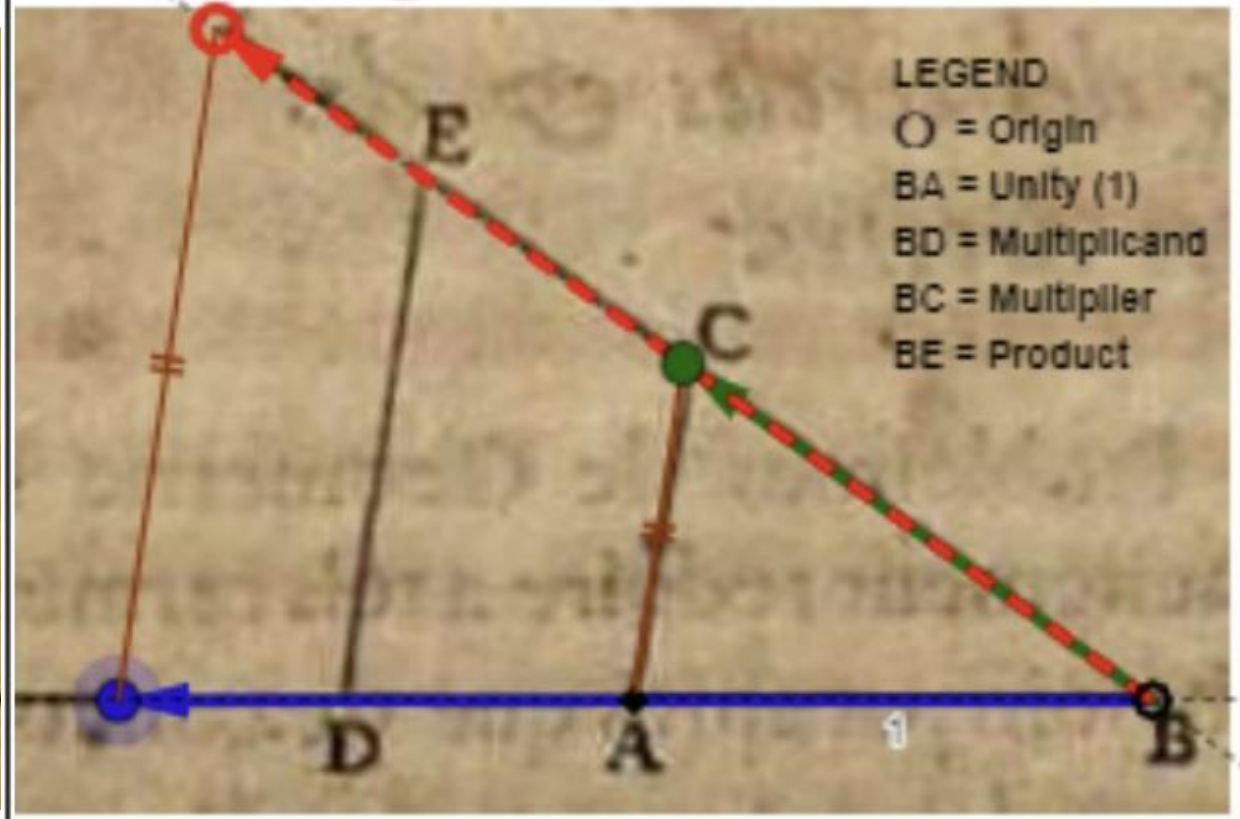
"For example, let BA be taken as Unity, (1), and let it be required to multiply BD (the Multiplicand) by BC (the Multiplier), I have only to join the points A and C, and draw DE parallel to AC; and BE is the Product of this Multiplication."

INSTRUCTIONS

Drag the Multiplicand (blue dot) and Multiplier (green dot) along the dashed lines or axes. Watch what happens to the red Product line when both the Multiplier and Multiplicand are negative (equal and opposite on the other side of the origin).

www.geogebra.org/m/edrukjbs

Indian Logic Meets Descartes' 1637 Multiplication Model www.podometic.i



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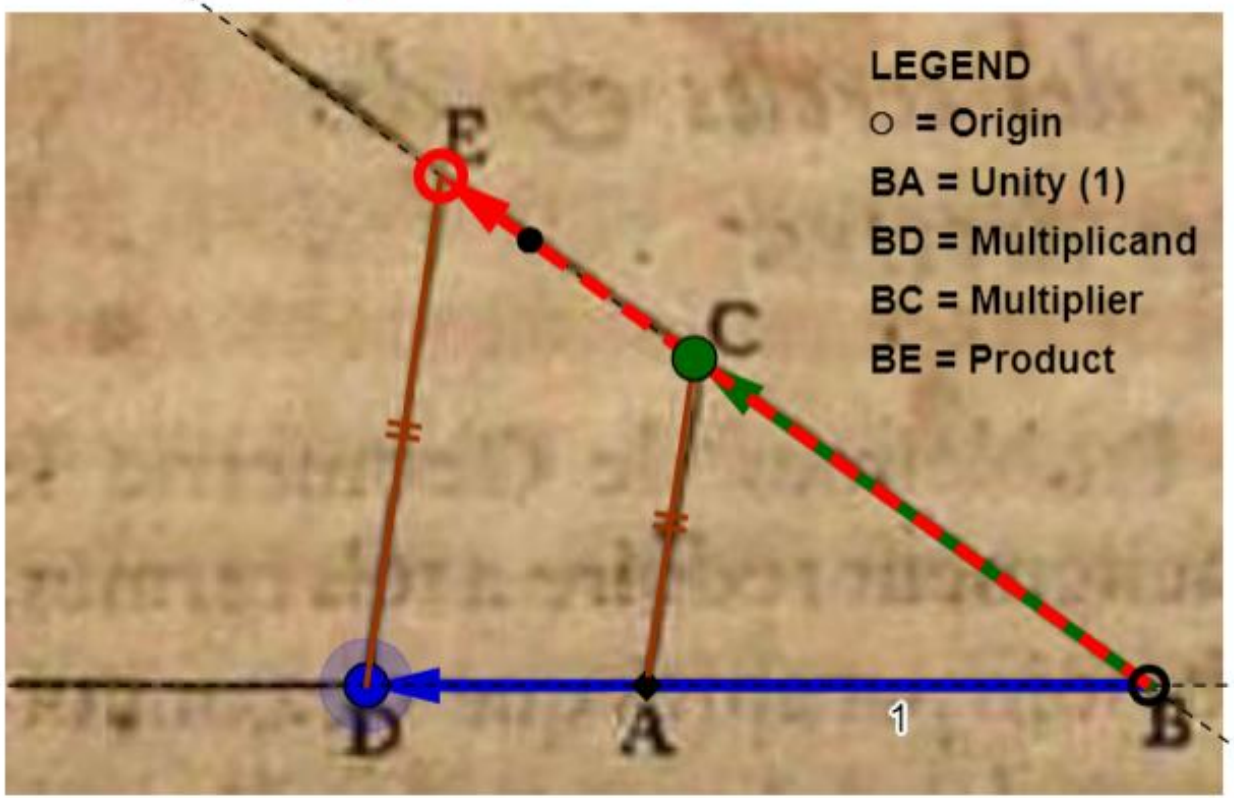

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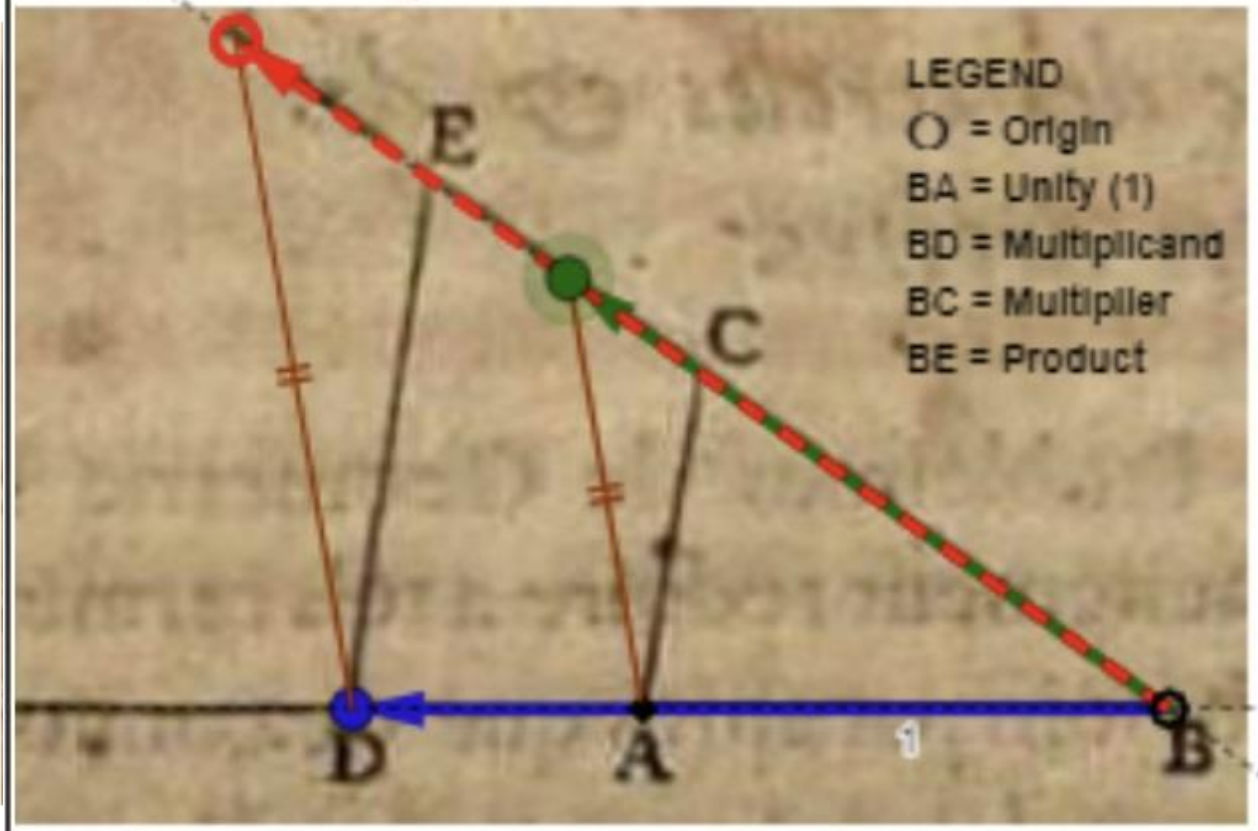

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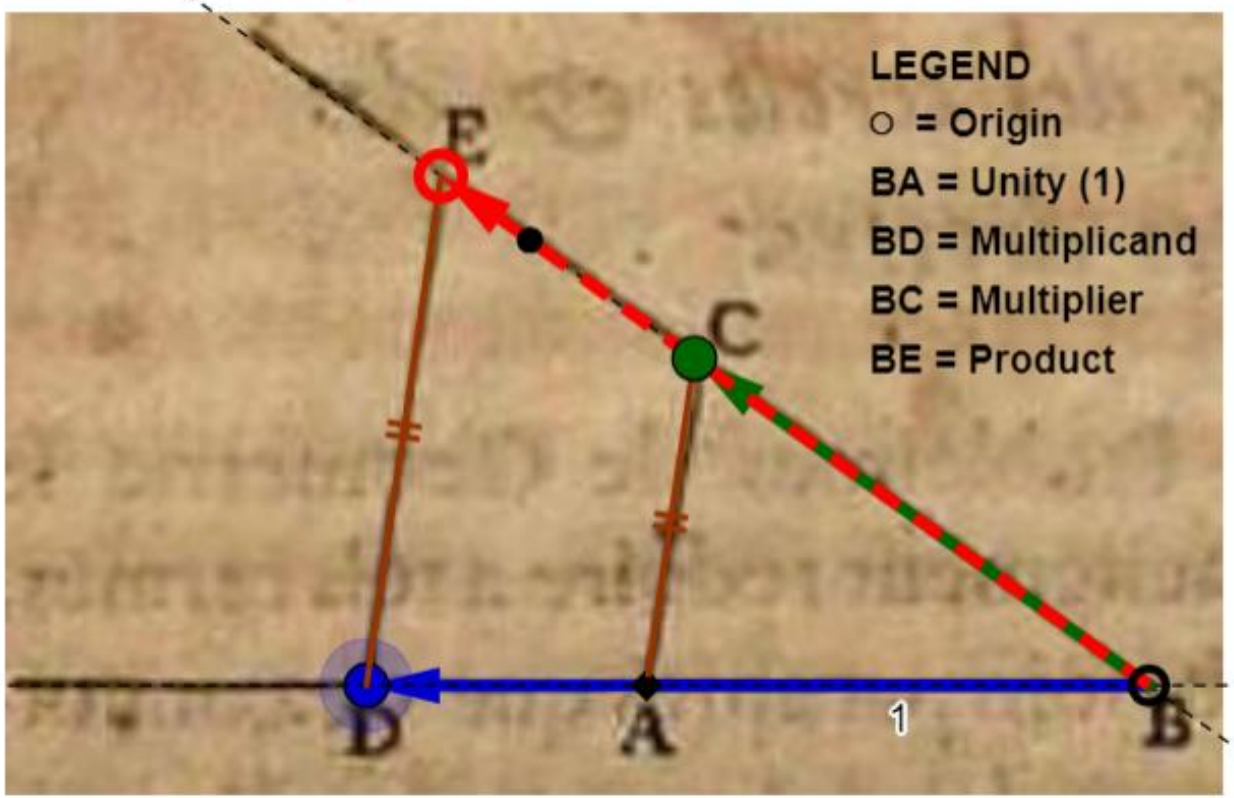

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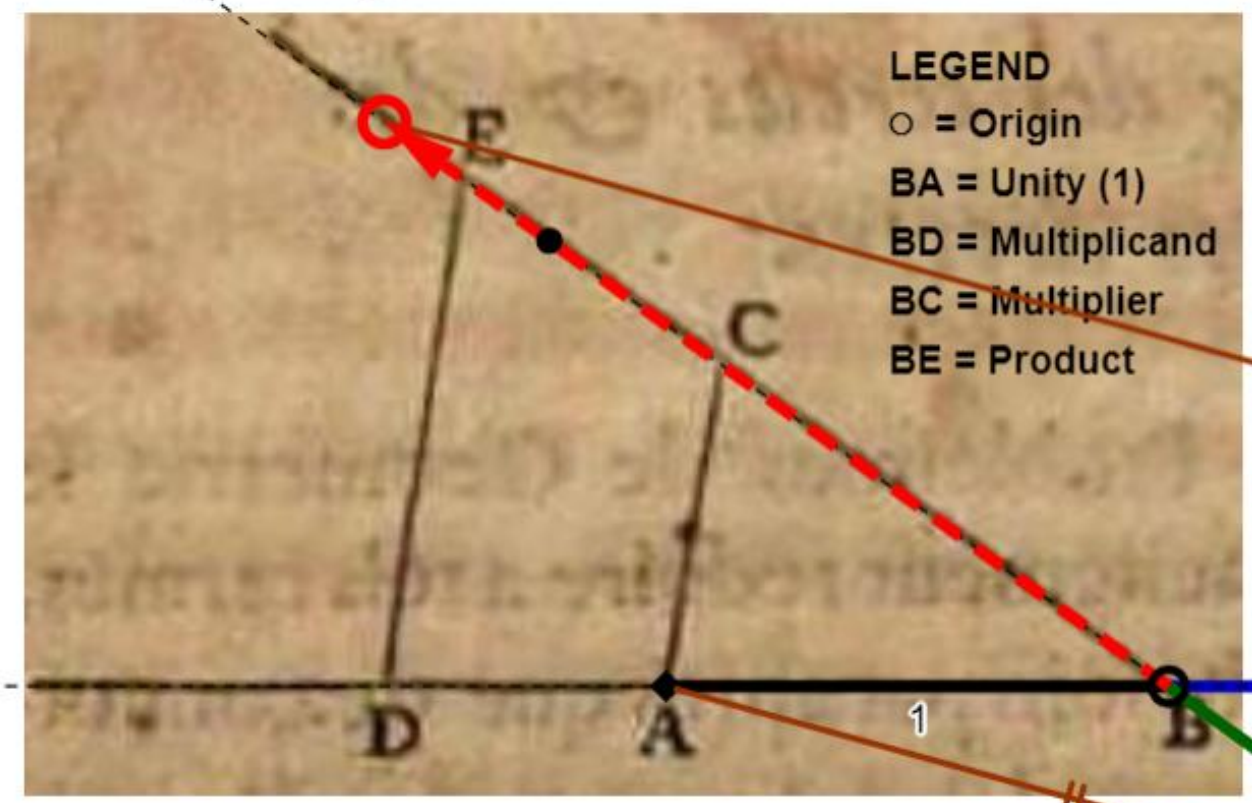
GeoGebra https://www.geogebra.org/m/wprjkwam

Indian Logic Meets Descartes' 1637 Multiplication Model.



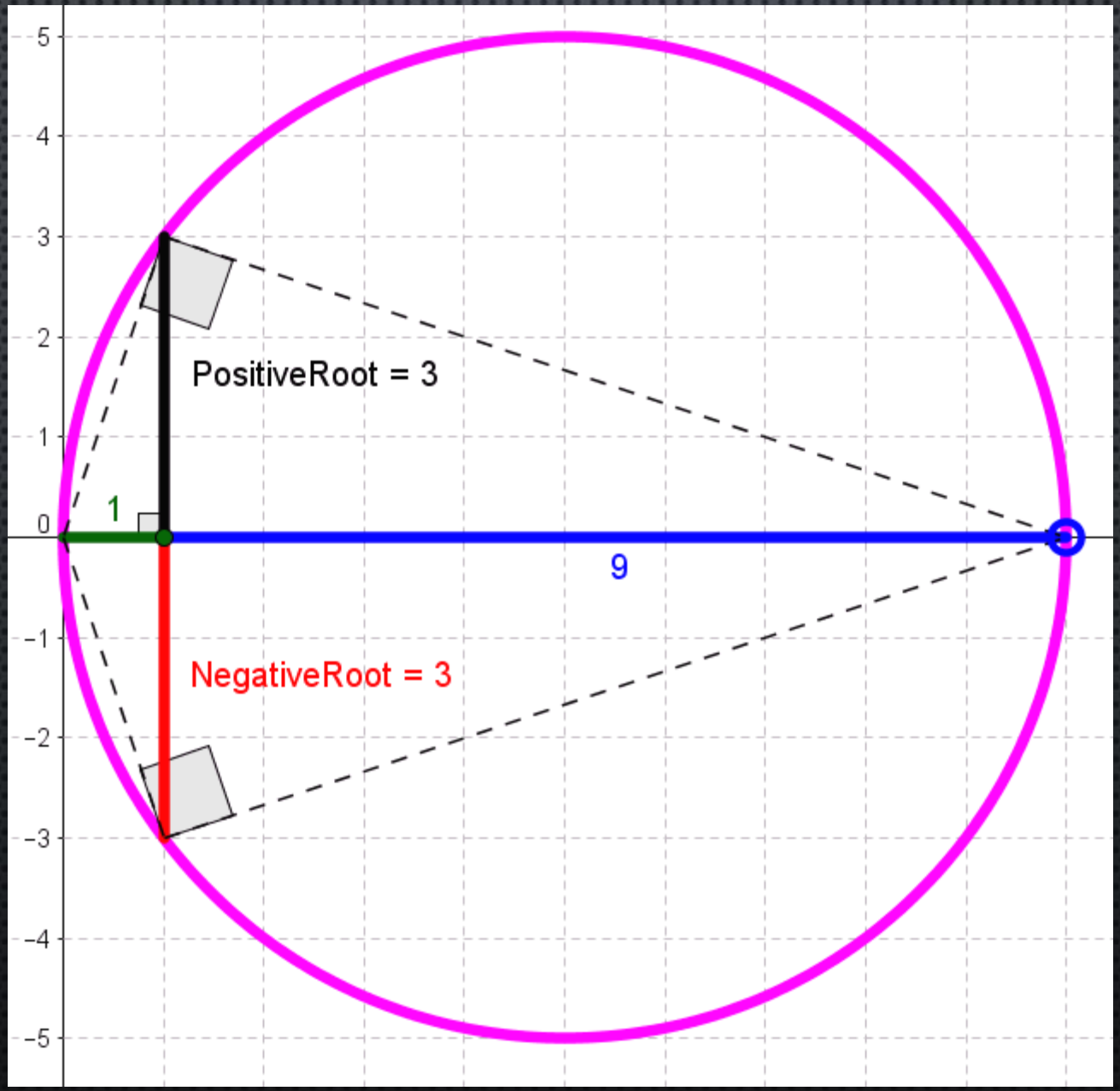
LEGEND
○ = Origin
BA = Unity (1)
BD = Multiplicand
BC = Multiplier
BE = Product

"For example, let BA be taken as Unity, (1), and let it be required to multiply BD (the Multiplicand) by BC (the Multiplier), I have only to join the points A and C, and draw DE parallel to AC; and BE is the Product of this Multiplication."

INSTRUCTIONS

Drag the Multiplicand (blue dot) and Multiplier (green dot) along the dashed lines or axes. Watch what happens to the red Product line when both the Multiplier and Multiplicand are negative (equal and opposite on the other side of the origin).

A Negative Multiplicand and a Negative Multiplier result in a Positive Product.


5
4
3
2
1
0
−1
−2
−3
−4
−5
PositiveRoot = 3
1
9
NegativeRoot = 3

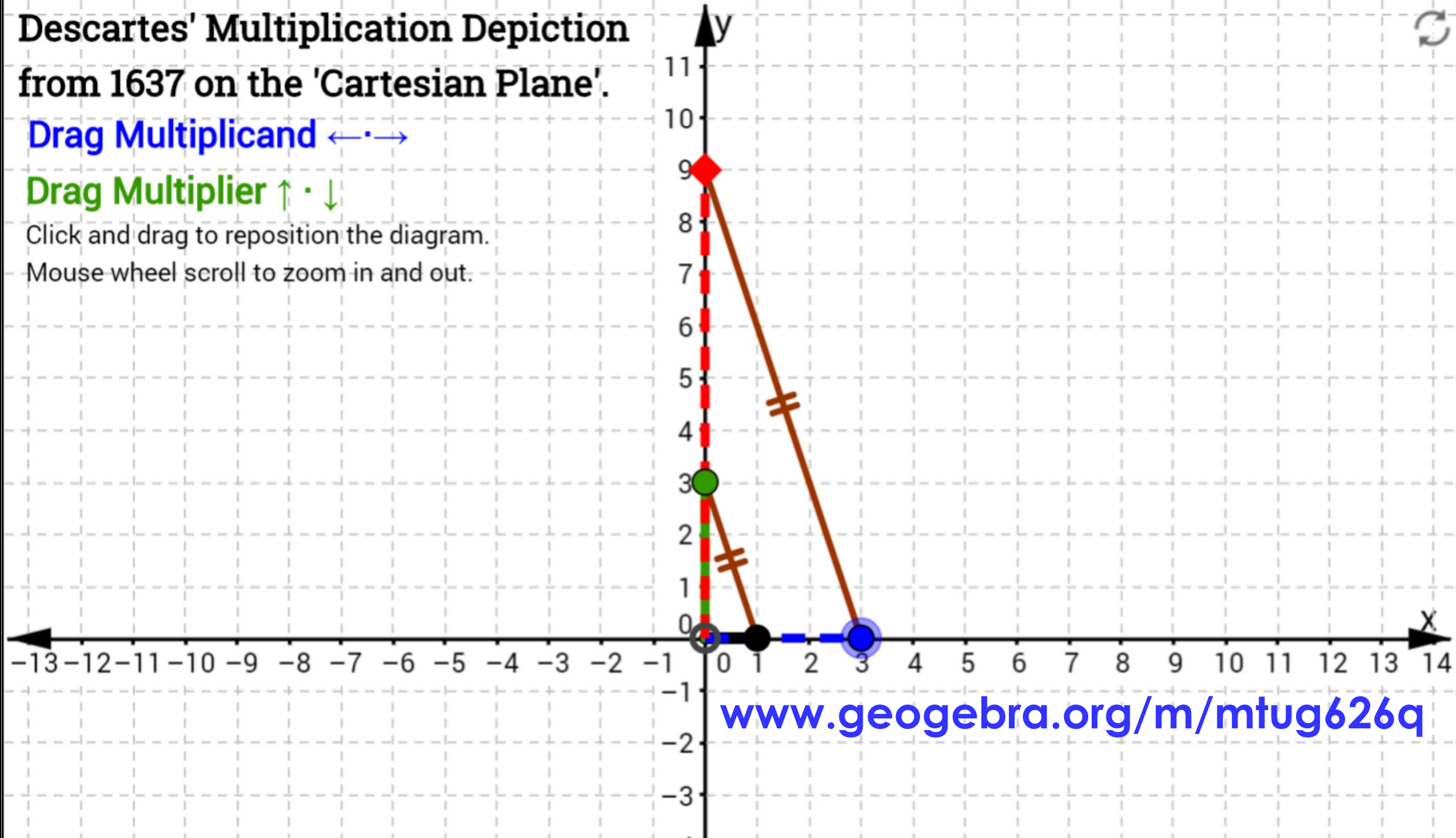
Descartes' Multiplication Depiction from 1637 on the 'Cartesian Plane'.

Drag Multiplicand ←·→

Drag Multiplier ↑ · ↓

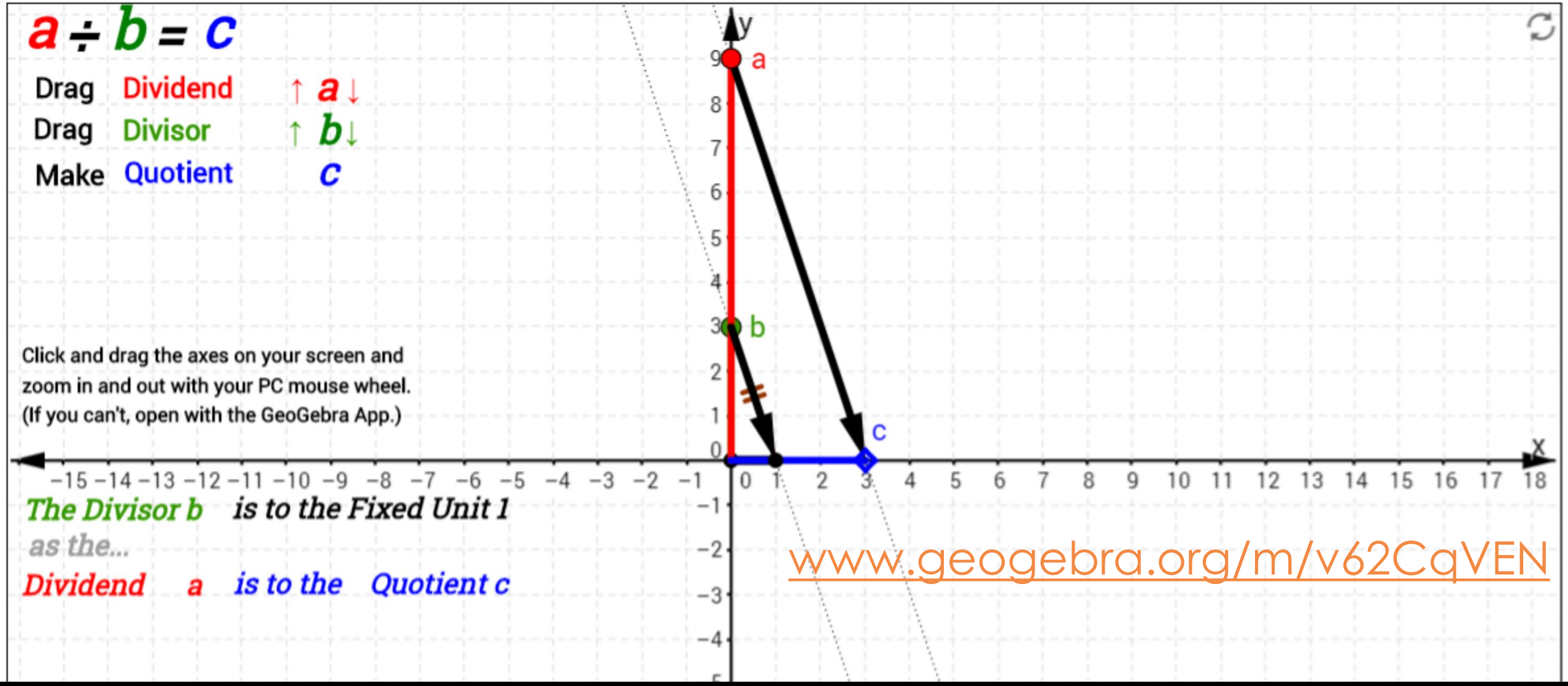
Click and drag to reposition the diagram.

Mouse wheel scroll to zoom in and out.



www.geogebra.org/m/mtug626q

DesCartesian Division



www.geogebra.org/m/v62CqVEN

THE CHILD'S OBSCURED MATHS MIND

Index Laws

Multiplication Makes More

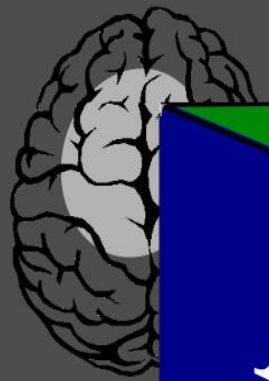
Negative Numbers

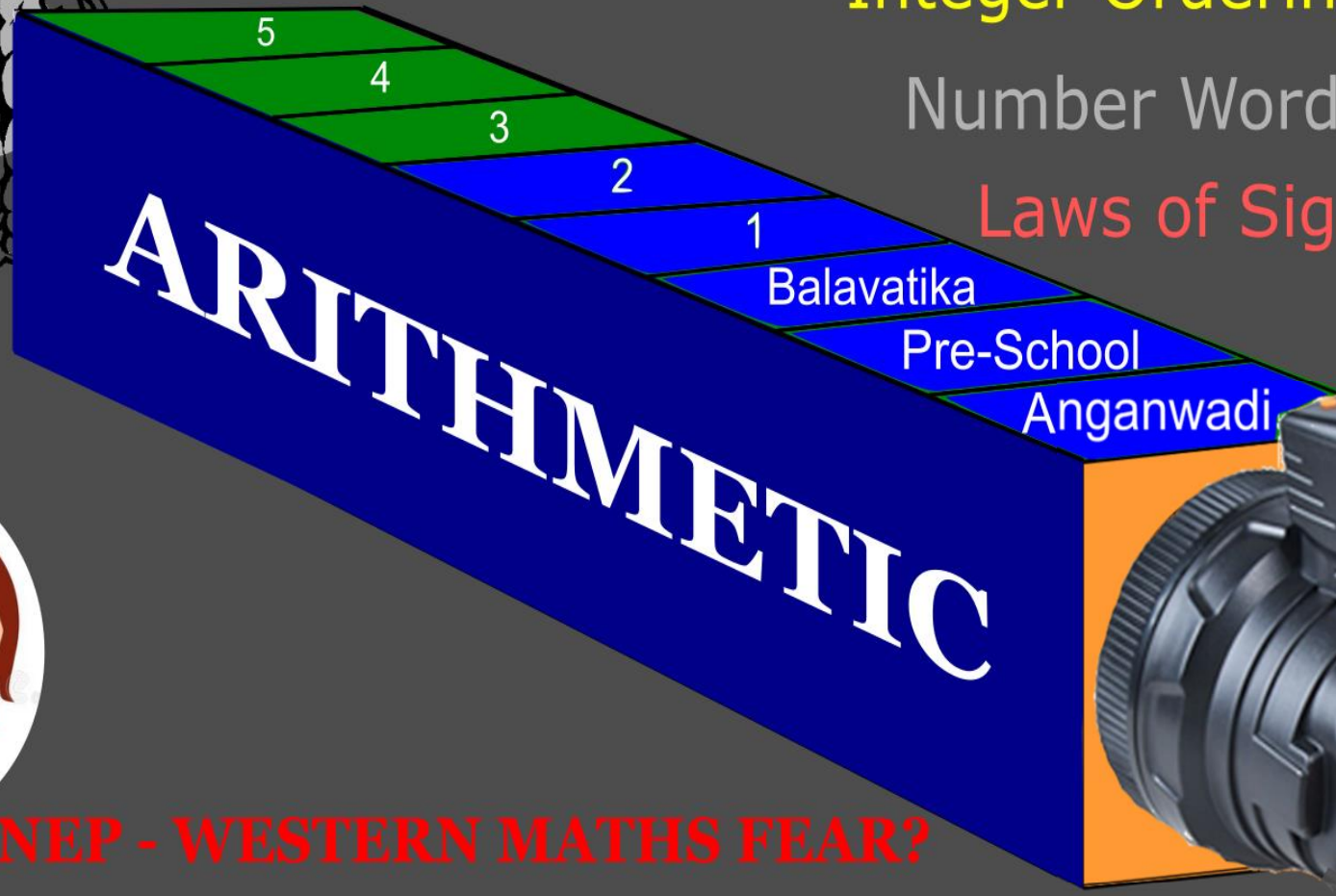
Rules not Reason

Integer Ordering

Number Words

Laws of Sign





NEP - WESTERN MATHS FEAR?

THE CHILD'S ILLUMINATED MATHS MIND

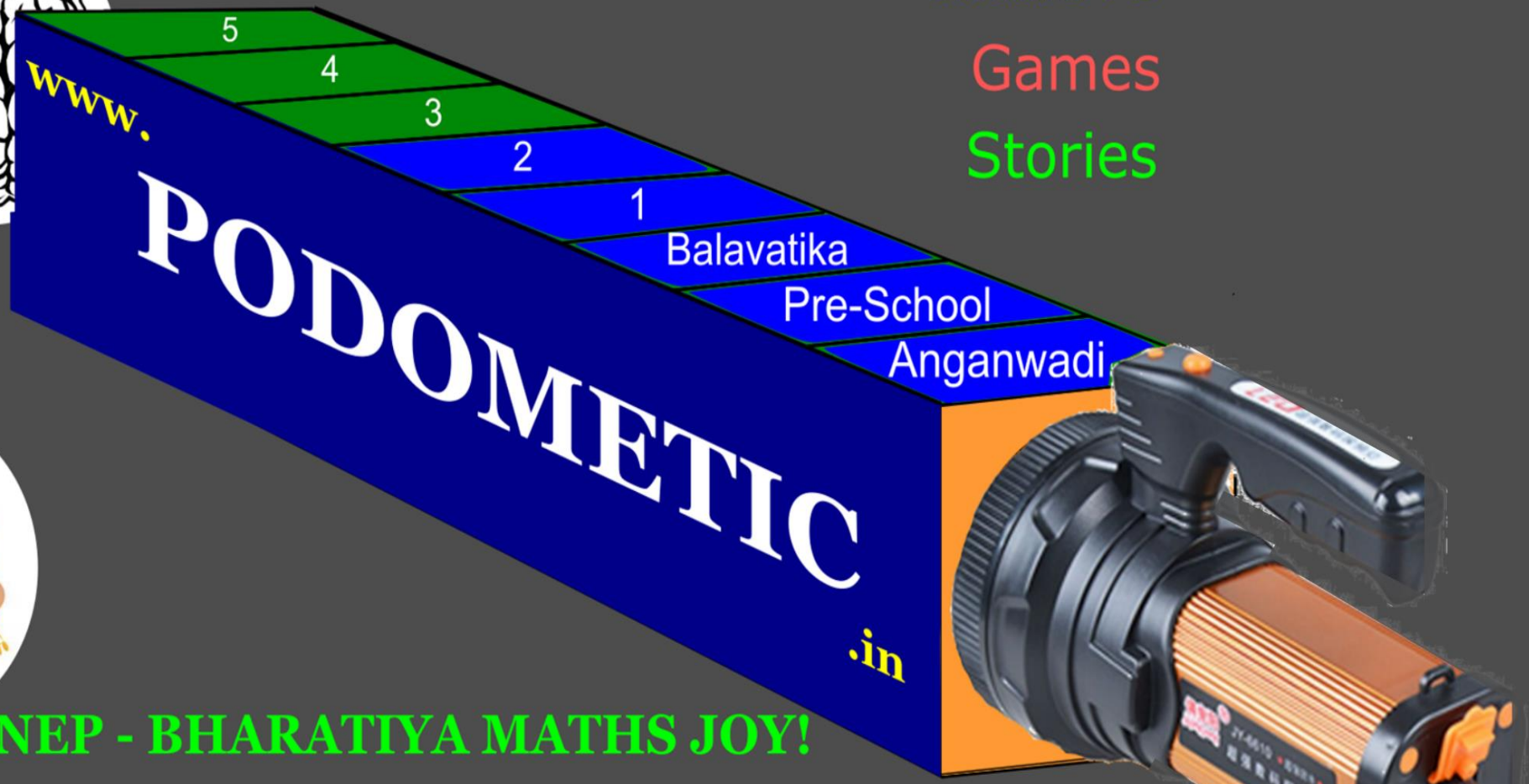
Common Sense

Multi-Sensory

Intuitive

Games

Stories



NEP - BHARATIYA MATHS JOY!

THANK YOU!

Brahmagupta's definition of zero failing to be transmitted to Europe via the Arabic world

Jonathan J. Crabtree www.podometic.in

INDIAN SOCIETY FOR HISTORY OF MATHEMATICS
DELHI INDIA | 22 DECEMBER 2020