Fun in Podo's Paddock: New Ways of Teaching Children the Laws of Sign for Multiplication and Division (and more!)

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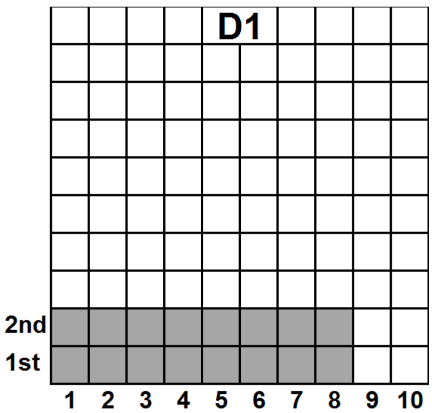
Abstract

Many teachers find it hard to explain the laws of sign of integers. Mathematics is a science of patterns, so $^{+}2 \times {}^{+}3 = {}^{+}6$ suggests $^{-}2 \times {}^{-}3 = {}^{-}6$, yet this is not so. Multiplicative pedagogies such as repeated addition, equal groups, arrays, and area models fail on the *Integers*. One cannot add a number of objects a negative number of times, nor have a negative number of groups, nor a negative number of rows, nor an area less than zero. Division has a repeated subtraction model and an equal shares model, yet without citing sign laws, teachers cannot explain how to solve $^{+}6 \div {}^{-}3$. There are no negative threes in positive six and you cannot divide six into negative three groups. So, I introduce new games in a play featuring a multi-cultural group of famous mathematicians as children. Informed by the original writings of great thinkers, I found and fixed enduring flaws in the teaching of elementary mathematics.

Introduction to Acts 1 & 2

In **Act 1**, day 1 of Grade 3, Teacher challenges Dio (Diophantus), and Gerry (Cardano), to explain (10 – 2) × (10 – 3). After some confusion, they succeed. In **Act 2**, Teacher challenges Dio and Gerry to solve (2 – 10) × (3 – 10). They can't. Yet, in Podo's Paddock, Eucy (Euclid), Huiey (Liu Hui), and Brammy (Brahmagupta) laugh as they play and solve the problem! Teacher, not believing the problem can be solved, taunts the children with division involving a positive dividend and a negative divisor. Teacher has not discussed either the laws of sign or negative numbers. However, Teacher is in for a shock!

Act 1



TEACHER: Dio, put 7 lots of 8 bricks on a table.

DIO: I place 8 bricks on the first row, then place 8 bricks on the second row… [**Fig. D1]**.

TEACHER: No, no, no! Too slow! How will you ever solve problems like (10 – 2) × (10 – 3)?

GERRY: Please explain that problem to us.

TEACHER: I thought you children were smart!

DIO: Wait, 8 bricks placed 7 times on a table is also said to be 8 bricks multiplied by 7. You disguised 8 bricks as 10 – 2 bricks and disguised the 7 times they are to be placed together as 10 – 3.

TEACHER: Indeed, we have (**10** – 2) × (**10** – 3). You must show me how you start with 100 bricks [**Fig. T1**] on your 10 × 10 table and finish with 56.

[DONG DONG]

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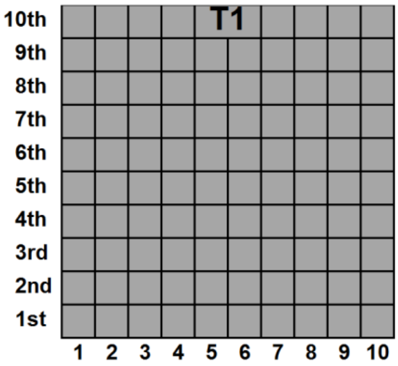
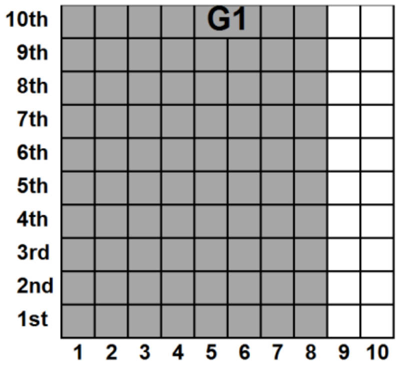
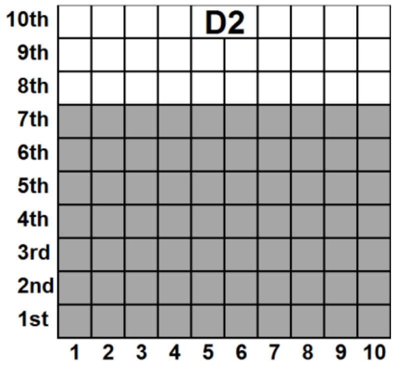
TEACHER: That's the bell. Everybody out but Dio and Gerry. *[Eucy, Huiey and Brammy exit with spades and square-side buckets to play in Podo's Paddock.]*

GERRY: I'm taking away the 2 extra bricks in each row of 10 [**Fig. G1**].

DIO: I'm taking away the extra 8^{th}, 9^{th} and 10^{th} rows of 10. [**Fig. D2**].

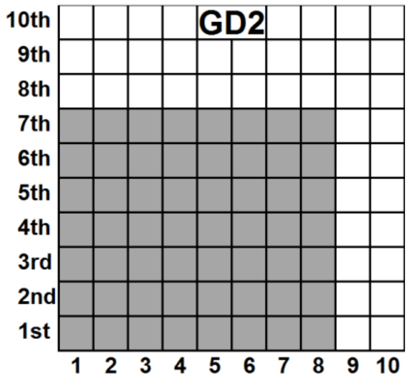






GERRY: So, I subtracted 20 bricks.

DIO: As I subtracted 30 bricks. So, on another table [**Fig. GD2**] we show 7 rows or lots of 8 bricks each, which is 56 bricks altogether!



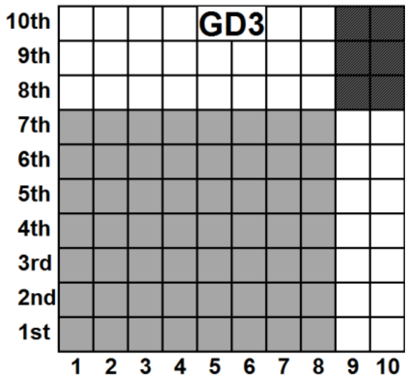
[Gerry & Dio frown as they check their work.]

GERRY: From 100, I took away 20 leaving 80.

DIO: While I took away 30, which leaves 50. We have 100 – 20 – 30 = 50, yet our work is meant to produce an answer of 56 bricks!

[Can you see where Dio and Gerry went wrong?]

GERRY: **You** went wrong! I subtracted 20 bricks up Eastside. So, you couldn't have subtracted 3 full rows of 10 bricks from Northside. Six bricks in the corner were already taken away!



DIO: No, **YOU** went wrong! After I subtracted 3 rows from Northside, you could not have subtracted 20 bricks from Eastside. I had already taken away the 6 bricks **YOU** claim to have subtracted [**Fig. GD3**].

GERRY: We can't take away the same 6 bricks twice. So – of course! We add back the extra 6 bricks **YOU** took away by mistake! Then, our equation becomes 100 – 20 – 30 $\underline{+\ 6}$ = 56.

AHA! *We have explored the equation (10 – 2) × (10 – 3) = 56, which may be written in the form* $(a - b) \times (c - d) = ac - bc - ad \underline{+\ bd}$. *Mathematicians DEFINE neg. × neg. = pos. to preserve the distributive property. Yet NOT ONE of these numbers is negative! In Arithmetica, (c. 250 CE), Diophantus wrote (in Greek), 'A wanting* [less 2] *multiplied by a wanting* [less 3] *makes a forthcoming'* [plus 6] (Heath, 1910). *The Greek of Diophantus appears in Latin as 'Minus per minus multiplicatum, producit Plus'* (Bachet, 1621). *Cardano* *initially accepted minus × minus = plus, yet later said minus × minus = minus, as the 6 bricks had been subtracted* (Cardano, 1663). *So, what happens next?*

Act 2

TEACHER: Minus by minus, a number gets plussed, the reason for this has now been discussed. Yet I challenge you to solve, not (10 – 2) × (10 – 3), but the problem of (2 – 10) × (3 – 10). If you are so smart, solve that!

GERRY: Teacher, it is one thing to start with 10 and take 2 away from it, yet to start with 2 and take 10 away from it is absurd! The first problem began with 100 bricks, from which I took away 20 while Dio took away 30.

DIO: The first step is 2 bricks placed 3 times, which produces 6 bricks. Neither 20 bricks nor 30 bricks can be subtracted from 6 bricks. It is impossible!

[From the classroom, the scene changes to kids playing in Podo's Paddock.]

EUCY: Teacher says two multiplied by three is two added to itself three times, yet I'd never say something that silly (Crabtree, 2016). My sister said 2 digits placed together 3 times produce 6 digits as do 3 digits placed together 2 times. Sadly, she was forbidden from showing Teacher her fingers as proof.

HUIEY: Your fingers must draw us squares Eucy, as: 'the dual natures of Yin and Yang sum up the fundamentals of mathematics' (Shen et al, 1999). We need bricks for Yang on our tables, yet all we have are spades and buckets.

BRAMMY: With our tools, we dig holes for Yin and make bricks for Yang!

HUIEY: Teacher says negative numbers are less than zero [the children laugh] yet never says less WHAT than zero! In China, such things were played with a thousand years before zero was used as a number. We used numbers of black rods for Yin (negative) things and red rods for Yang (positive) things.

BRAMMY: Negative numbers are less *positive* than 0. Positive numbers are less *negative* than 0. A hole that is 1 meter deep is less *high* than ground level 0. Yet a brick that is 1 meter high is less *deep* than ground level 0.

[Gerry has snuck out to join the gang.]

EUCY: Aha! Now I see the equal measure [sym-metry]. Numbers can be the same, while their units are equal and opposite in nature.

GERRY: I can combine two lots of same things. Yet how do we join or combine equal and opposite things, like bricks and holes? Teacher told Dio and me to solve the problem (2 – 10) × (3 – 10).

HUIEY: When two soldiers from opposite sides join in battle, they both die! So, whatever side has more soldiers wins! Should an Eastern army with 24 soldiers battle a Western army with 13 soldiers...

GERRY: … the Eastern side wins with 11 Eastern soldiers remaining!

BRAMMY: In the West, opposites are more likely said to be: boy-girl, big-little, soft-hard, narrow-wide, fast-slow, clean-dirty, cheap-expensive, long-short, strong-weak, rough-smooth, sick-healthy, new-old and so on.

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HUIEY: We had ideas like that in the East, yet when it came to mathematics, our ideas of opposites were just things that cancelled each other out.

BRAMMY: Teacher says 0 is a number subtracted from itself, $\boldsymbol{n} - \boldsymbol{n} = 0$. Yet I say zero is the sum of equal and opposite numbers of equal size (Dvivedī, 1902) written as $^{-}\boldsymbol{n}$ *&* $^{+}\boldsymbol{n} = 0$*. Take away* $^{-}\boldsymbol{n}$ from $^{-}\boldsymbol{n}$ *&* $^{+}\boldsymbol{n}$, or zero and $^{+}\boldsymbol{n}$ remains. *Take away* $^{+}\boldsymbol{n}$ from $^{-}\boldsymbol{n}$ *&* $^{+}\boldsymbol{n}$, or zero and $^{-}\boldsymbol{n}$ remains.

GERRY: That's why, if from 5 black rods Huiey takes away 7 black rods, his answer is 2 red rods. Once he has subtracted 5 black rods from 5 black rods he has nothing, which as zero, equals 2 red and 2 black. So, when he takes away the final 2 black rods, the 2 red rods remain. The Chinese are smart!

BRAMMY: China had negative numbers and positive numbers, yet not as westerners think. Ask: *'What is negative seven minus negative four?'* and most say negative eleven. Yet eastern children knew 7 negatives minus 4 negatives is 3 negatives. Westerners are confused by negatives because they have the wrong definition of zero, which India put right.

HUIEY: Any math object, taken away from Brammy's zero becomes its opposite. So, 2 bricks take away 10 bricks may be thought of as 2 bricks take away (2 + 8) bricks. After we have taken away 2 bricks from 2 bricks, we have 0 bricks on ground level zero. So, we split 0 by rebuilding it as 8 holes and 8 bricks. Then when we take away 8 bricks, 8 holes remain!

EUCY: Westerners got numbers via Greek geometry which did not have ideas of zero, positives and negatives. As there are four of us, I've drawn four tables. Now, we must multi-play to solve the problem (2 – 10) × (3 – 10).

BRAMMY: The multiplicand (2 bricks – 10 bricks) is our number of things to be multiplied and this is 8 holes. The multiplier (3 – 10) tells us how many times our multiplicand is to be added to or subtracted from zero. Obviously, when 3 additions battle with 10 subtractions, the subtractions win and 7 remain!

HUIEY: A Hole and a Brick of equal size make Zero when the Brick and Hole are combined [H & B = Z]. When a Brick is taken away from Zero a Hole remains and when a Hole is taken away from Zero a Brick remains.

BRAMMY: Since we started with bricks, (2 – 10) × (3 – 10) is like 8 holes subtracted from ground level zero 7 times. To take away 8 holes you add 8 bricks. Adding 8 bricks 7 times gives you 56 bricks! Let me double check. From (2 – 10) × (3 – 10), we have (8 holes) taken or placed together (3 times minus 10 times). Eight holes placed 3 times means 24 holes are added to zero. Eight holes taken away 10 times means 80 bricks are added to zero. As 24 holes cancel out 24 bricks, 56 bricks remain. It is correct!

GERRY: Teacher says we must use tables, not common sense.

EUCY: Fine. We have buckets, spades and four tables so let's multi-play!

HUIEY: Math is a game to play and our mind is the playground! Westside is for hole-play and Eastside for brick-play. Northside counts the number of times holes or bricks are Added to zero [$^{-}\boldsymbol{n}$ & $^{+}\boldsymbol{n}$] and Southside counts the number of times holes or bricks are Subtracted from zero. Some confuse

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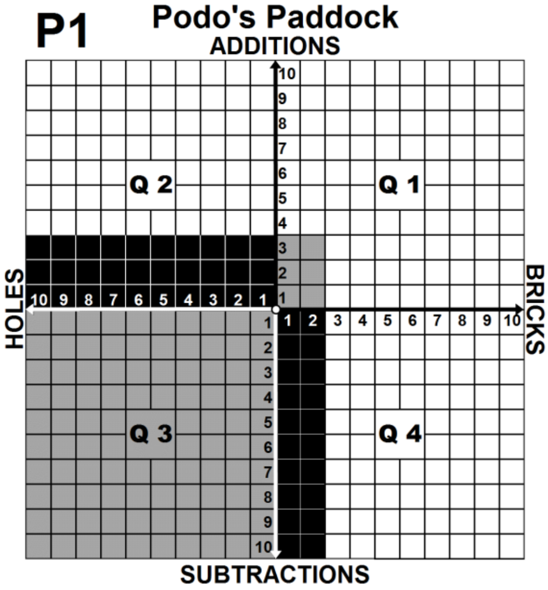
subtraction and addition with negative and positive so holes and bricks are marked with the mini-signs $^{-}$ and $^{+}$ instead of – and +. So, let's multi-play (2 – 10) × (3 – 10).



BRAMMY: For 2 bricks and 10 holes we write ($^{+}$**2** & $^{-}$10). The multiplier telling us to add to or subtract from ground level zero appears as (**0 + 3** & 0 – 10). So, 2 bricks added 3 times onto zero makes 6 bricks in **Q1** **[Fig. P1]**.

HUIEY: Then we play ($^{+}$**2** & $^{-}$10) × (0 + 3 & **0 – 10**). This gives us 2 bricks subtracted 10 times from zero, which makes 20 holes in **Q4**.

BRAMMY: After that, we play ($^{+}$2 & $^{-}$**10**) × (**0 + 3** & 0 – 10). So, 10 holes added 3 times onto zero makes 30 holes in **Q2.**

EUCY: I see. We then play ($^{+}$2 & $^{-}$**10**) × (0 + 3 & **0 – 10**). This is 10 holes subtracted 10 times from zero, which makes 100 bricks in **Q3.**

Q1. $^{+}$2 × (0 + 3)	**Q2**. $^{-}$10 × (0 + 3)	**Q3**. $^{-}$10 × (0 – 10)	**Q4**. $^{+}$2 × (0 – 10)
2 Bricks Added 3 Times onto Zero makes 6 Bricks	10 Holes Added 3 Times onto Zero makes 30 Holes	10 Holes Subtracted 10 Times from Zero makes 100 Bricks	2 Bricks Subtracted 10 Times from Zero makes 20 Holes

BRAMMY: So, 6 bricks in **Q1** and 100 bricks in **Q3** is 106 bricks.

HUIEY: And 30 holes in **Q2** and 20 holes in **Q4** is 50 holes. When 106 bricks battle with 50 holes, the bricks will win and 56 bricks will remain.

BRAMMY: We must fetch Dio so he can see the wonder of negative and positive. We must fetch Teacher so he can see the impossibility of quantities less than zero. The impossible problem of (2 – 10) × (3 – 10) on Podo's paddock, has 100 – 30 – 20 + 6 and that produces $^{+}$56 in **Q3**. **What fun!**

HUIEY: With the ebb and flow of Yin and Yang, what is Product becomes Dividend, while Multiplicand and Multiplier become Quotient and Divisor.

BRAMMY: Positive products and dividends are in **Q1** by addition or **Q3** by subtraction. Negative products and dividends are in **Q2** by addition or **Q4** by subtraction. A positive dividend with a subtractive divisor needs negatives.

TEACHER: Playtime is OVER! This will spoil your fun. What's 56 ÷ $^{-}$7?

ALL: **Eight holes!**

[DONG DONG]

Introduction to Acts 3 & 4

In **Act 3** a year has passed. It is day 1 of Grade 4. Teacher banishes the gang to a Grade 2 classroom. That's where the gang teams up with Roomy (Adriaan van Roomen) and Carty (René Descartes) who teach as much as they are taught! Explanations of *Integer* multiplication and division emerge in a way that Teacher was never taught. In **Act 4** the new gang heads out to play in Podo's Paddock where they meet Wally (John Wallis), Zacky (Isaac Newton) and Arcy (Archimedes). They soon have fun multiplying and dividing not just *Integers*, but what today are called *Real* numbers!

Act 3

TEACHER: Good morning class.

ALL: Good - morning - Teacher.

TEACHER: Next year you'll learn the 'Golden Rule'. Who knows what that is?

BRAMMY: Whoever has the most gold makes the rules!

[The gang all laugh at Brammy's joke.]

TEACHER: STOP LAUGHING! Brammy and all who laughed will not enjoy my lesson! Instead, you must head straight to Grade 2C as their teacher is sick today. YOU will be teachers and YOU will teach mathematics!

[The gang walks to Grade 2C, happy to be able to have fun in a classroom.]

GERRY: My mother said the Golden Rule, like multiplication, is a way to find the fourth number with three given numbers. She called it the Rule of Three.

HUIEY: My sister sells fruit. She says by knowing three numbers, the fourth is found. She sells 3 golden delicious apples from our tree for 12 cents. If asked the price of 5 apples, she applies the Rule of Three. She says as 3 is to 12 so 5 is to price. All she does is multiply 12 × 5 to get 60 and divide by 3 to get 20. So, the price of 5 apples is 20 cents. I have many for us to eat!

BRAMMY: Why doesn't she just sell the golden apples at 4 cents each?

HUIEY: I asked that. She said, 'Why sell one apple when you can sell three?'

BRAMMY: Girls are smart! Patty [Hypatia] wrote about Dio and Eucy's ideas!

GERRY: How will the children find the fourth number after being given three numbers? They may only know **direct variation** (DV) for addition and subtraction and how to find the third number after being given two numbers.

EUCY: We will use bricks. To keep their interest, we must make it a game.

[The gang enters Grade 2C. With no teacher, the kids are running about.]

HUIEY: Quiet! Sit down! We have golden apples. Today, we make the rules!

BRAMMY: I see you all have a set of bricks. Can you count? Watch me as I draw one dot on one side of a wooden block. OK, what comes after one?

ROOMY: The number two comes after the number one.

BRAMMY: Then I will draw two dots on the second side of my block, which will become our dice. So, what number comes after two?

CARTY: The next number is three!

[Before long, each side of the dice has a unique number of dots on it.]

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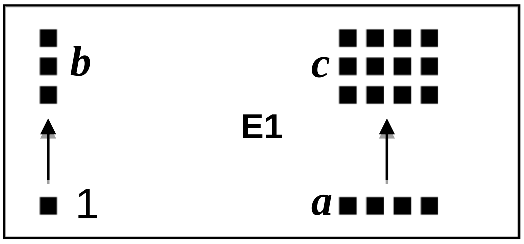
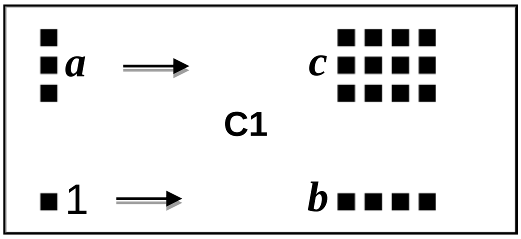
BRAMMY: Place a single brick close to you. That *Unit* remains fixed while we play. I will now roll my dice on the floor. Whatever number of dots is shown atop it when it lands will be our *Multiplier* ***b***.

[Brammy rolls the dice and it stops with three dots on the top.]

BRAMMY: For each dot, place a brick away from you, so ***b*** has 3 bricks. Now we roll for a number to be multiplied, the *Multiplicand* ***a***, and we have… Four! So, place 4 bricks close to you, as *Multiplicand* ***a***.



EUCY: Math will forever confuse unless you know this! From three numbers, the **Unit** (**1**), **Multiplicand** (***a***) and **Multiplier** (***b***), we find the fourth number, **Product** (***c***). The Golden Apple Rule is this. ***Whatever you do to a Unit to make the Multiplier, you do to a Multiplicand to make the Product.*** Just as you place *Units* to make a *Multiplier*, you place *Multiplicands* to make a *Product* [**Fig. E1**]. Whatever **1** does to make a *Multiplier*, the *Multiplicand* must do to make the *Product!* As I sit at Southside, we start with the *Unit* (**1**) close and in our mind, place it three times far. So, I have the *Multiplier* 3 at ***b*** shown by three bricks. Then, I have to multiply four bricks, so the *Multiplicand* at ***a*** is 4. So, it is also to be placed three times to make the *Product* at ***c***.



CARTY: Yet as I sit at Westside. I can start with the *Unit* (**1**) close to me and place it four times far from me, to have the *Multiplier* 4 at ***b*** shown by four bricks [**Fig. C1**]. Then, I have to multiply three bricks, so that means the *Multiplicand* 3 at ***a*** is to be placed four times far from me to make the same *Product* at ***c***. Our three given numbers are the *Unit* (1) and two input factors, the *Multiplicand* ***a*** = 3 and the *Multiplier* ***b*** = 4. **As 1 is to *b*, so *a* is to *c*!** Eucy and I have different *Multipliers* and *Multiplicands* yet get the same *Product!* So, either factor may be placed as many times as the other factor has units! Let us multiply the *Multiplicand* ***a*** by the *Multiplier* ***b***, which we write ***a*** × ***b***. First, we have ***a*** = 4 and ***b*** = 3 [**Fig. E1**]. In ***a*** × ***b*** = ***c*** we place ***a*** together ***b*** times. So, 4 bricks placed together 3 times make 12 bricks. Alternatively, [**Fig. C1**] we have ***a*** = 3 and ***b*** = 4 and 3 bricks placed together 4 times also make 12 bricks. **1 is to *b*** the same way ***a* is to *c*** [i.e. **1 : *b* = *a* : *c***].

EUCY: As we have seen, I say 'if four numbers are proportional, then they are also proportional alternately' (Heath, 1908). [NOTE: From 1 × ***c*** = ***b*** × ***a*** and **1 : *b* = *a* : *c*** we have 1 **: *a* = *b* : *c*** and 1 × ***c*** = ***a*** × ***b***. So, via Euclid and proportion, we prove the Commutative Law, as 1 × ***c*** = ***b*** × ***a*** = ***a*** × ***b***.]

ROOMY: *[♫ ♪♪ ♫ Singing to a tune like Baa Baa Black Sheep]* Multi-plying can be fun you ***c*.** Do to ***a*** as 1 made ***b***.

HUIEY: That's a great song. Where can I get the music? Now consider $^{+}2 \times {}^{-}3$. Restating the *Multiplier* as (0 – 3) means the *Multiplicand* $^{+}2$ must be

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subtracted from zero three times in succession. So, $^{+}2$ subtracted from zero three times means we have 0 – $^{+}2$ – $^{+}2$ – $^{+}2$ which then equals 0 – $^{+}6$ or $^{-}6$. Remember, any number subtracted from zero becomes its opposite. Now, we can apply the Golden Apple Rule, ***whatever you do to a Unit to make the Multiplier, you do to a Multiplicand to make the Product.*** So, what did we do to the *Unit* $^{+}1$ to make the *Multiplier* $^{-}3$?

CARTY: We took 3 *Units* and subtracted them from 0 to make the *Multiplier* $^{-}3$.

HUIEY: We took 3 *Units* and changed their sign from Yang to Yin. So, having DONE THAT, we DO THIS! We take 3 Yang *Multiplicands* and change their sign to make a Yin *Product.* Now, positive two multiplied by negative three $^{+}2 \times {}^{-}3$ makes us ask, ‘What did we do to 1 to make $^{-}3$?’ The answer is, we trebled it and changed its sign. Having DONE THAT, we DO THIS! We treble the *Multiplicand* $^{+}2$ and change its sign to make the *Product* $^{-}6$.

CARTY: Why do you say Yin and Yang instead of negative and positive?

BRAMMY: In the West, churches had a problem with ideas such as infinity and zero, whereas India did not. Western Churches controlled not only society, but education in society and only God was infinite. The void, or zero, was the realm of the devil. Somehow, only God was positive and infinite, while the devil, as zero or less than zero, was negative.

HUIEY: Yin and Yang are opposite and complementary in nature. That is the Tao, or way of mathematics. Just as 3 left is not worse than 3 right, Yin numbers are not worse than Yang numbers. Numbers of negatives are not always undesirable, just as numbers of positives are not always desirable. My uncle was tested and got a positive result, which was bad and made us sad. Then six months passed and he was retested and got a negative result, which was great and made us happy!

ROOMY: Was it a mathematics test?

HUIEY: No. It was medical. He no longer tests positive for a disease.

CARTY: Let’s sing! ♫ ♪ ♪ ♫ Division you ***c*** is lots of fun. Do to ***a*** as ***b*** made 1.

HUIEY: In the East, we learn a nine nines song, yet I haven’t heard this rhyme.

BRAMMY: Please explain ‘Division you ***c*** is lots of fun. Do to ***a*** as ***b*** made 1’.

CARTY: In division, we have ***a*** ÷ ***b*** = ***c***. The *Dividend* ***a*** is to be divided by the *Divisor* ***b***, and we must find the *Quotient* ***c.*** Singing *‘Division you’ll **c** is lots of fun. Do to **a** as **b** made* 1*’* tells us how to solve the problem.

ROOMY: From the *Unit* 1 we make a *Multiplier* ***b***. Yet from the *Divisor* ***b*** we make a *Unit* 1. From the *Divisor* of 3, we took one of three equal parts to make 1. So, having DONE THAT, we DO THIS! We take one of three equal parts of the *Dividend* ***a*** to make the *Quotient* ***c.*** More important than seeing patterns alone, is seeing ratios or *relationships*. As Eucy says, ‘a *ratio* is a sort of *relation* in respect of size between two magnitudes [or quantities] of the same kind’, (Heath, 1908). In multiplication, as **1** is to ***b***, so ***a*** is to ***c***. In division, as ***b*** is to **1**, so ***a*** is to ***c***. In verse, we sing to multiply and inverse we

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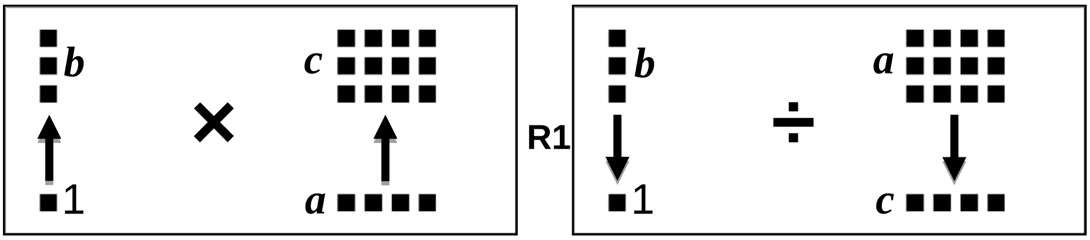
sing to divide. Between the FOUR terms, arrows go one way to multiply and the opposite way to divide [**Fig. R1**]. To multiply and divide, from three terms, we must find the missing fourth term to make the relationships or ratios equal. Two equal ratios make a proportion and it’s all a game! In multiplication, as 1 is to ***b***, given ***a***, what will we ***c?*** In division, as ***b*** is to **1**, given ***a***, what will we ***c?*** Math is our playground making us sing with joy!



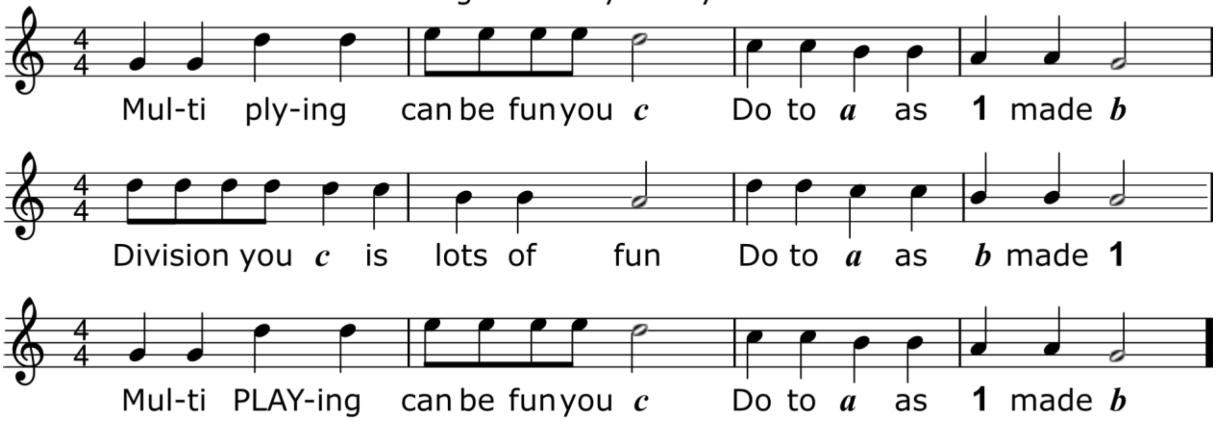
R1

The Multi-Play Song!

$a \times b = c$

$a \div b = c$

Traditional Arrangement. Lyrics by Jonathan Crabtree.

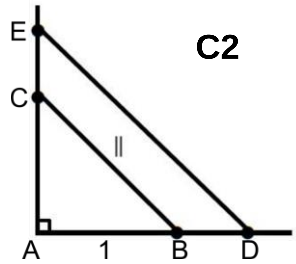


Act 4

GERRY: We have multiplied with squares, so let’s try multiplying with triangles. Then, whoever discovers how to multiply with circles wins the J. C. Paddocks Medal. If I were a gambler, I’d bet Arcy will get his name on that prize.

ARCY: Squares and triangles are fun. But I like arcs and circles the most!

CARTY: ‘Let AB be taken as unity and let it be required to multiply AD by AC, then I have only to join the points B and C, and draw DE parallel to BC; and AE is the product of this Multiplication’ (Descartes, 1637).



C2

EUCY: Here we have four terms, the *Unit* AB, the *Multiplicand* AD, the *Multiplier* AC and the *Product* AE. The triangles CAB and EAD are similar, so as the *Unit* is to the *Multiplier*, the *Multiplicand* is to the *Product* [**Fig. C2**].

HUIEY: That is nice. Yet we no longer have ideas of Yin and Yang or negative and positive. Do any westerners have a way of explaining opposites?

ZACKY: Yes! Wally wrote notes on that. Where’s Wally?

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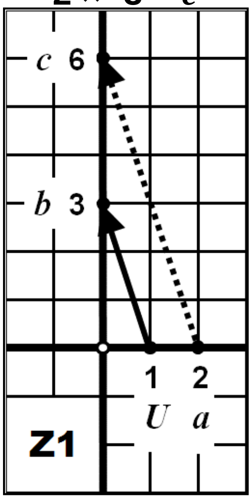
WALLY: 'The true notion of Multiplication is this, to put the *Multiplicand*, or thing multiplied (whatever it be) so often, as are the *Units* in the *Multiplier*; and consequently, if the *Multiplier* be more than 1, (suppose 2,) the *Multiplicand* is to be put more than once, (suppose twice,) and is therefore increased: If the *Multiplier* be 1, the *Multiplicand* is put just once, and therefore neither increased nor diminished' (Wallis, 1685).

HUIEY: We agree. A positive *Multiplier* leads to addition of the *Multiplicand* to zero and a negative *Multiplier* leads to subtraction of the *Multiplicand* from zero. What of the opposites of mathematics? How do you explain our Yin and Yang which are your negative and positive?

WALLY: I will update what I wrote. 'In case the multiplier is a negative number; suppose $^{-}2$; then instead of Adding the multiplicand to zero 2 times, it will signify so many times to Subtract the multiplicand from zero. For as $\boldsymbol{a} \times {}^{+}2$ implies twice adding $\boldsymbol{a}$ to zero; $0 + \boldsymbol{a} + \boldsymbol{a}$, to arrive at $^{+}2\boldsymbol{a}$, so $\boldsymbol{a} \times {}^{-}2$ implies twice subtracting $\boldsymbol{a}$ from zero; $0 - \boldsymbol{a} - \boldsymbol{a}$, to arrive at $^{-}2\boldsymbol{a}$. If + signify Upward, Forward, Gain, Increase, Above, Before, Addition, etc., then –, is to be interpreted of Downward, Backward, Loss, Decrease, Below, Behind, Subtraction, etc. And if + be understood of these, then – is to be interpreted of the contrary' (Wallis, 1685).

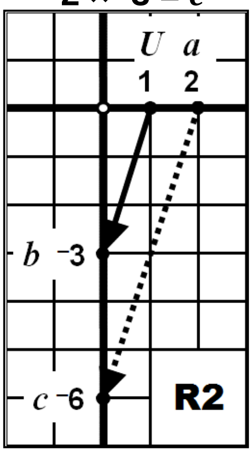
$^{+}2 \times {}^{+}3 = c$



ZACKY: 'To every action there is always opposed an equal reaction' (Newton, 1729). Let us multi-play either side of zero, because, 'if a line drawn a certain way be reckoned for positive, then a line drawn the contrary way may be taken for negative' (Newton, 1720). Based on what Eucy and Carty have said, we draw the multiplication 2 × 3 as follows. We draw a solid line from our fixed *Unit* 1 to the *Multiplier* 3, then draw a dashed parallel line from the *Multiplicand* 2, and it lands on the *Product* 6 [**Fig. Z1**]. So, again, we see how, for $\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{c}$, as **1** is to $\boldsymbol{b}$, so $\boldsymbol{a}$ is to $\boldsymbol{c}$. The *Unit* and *Multiplicand* vary proportionally.

ROOMY: Multi-playing can be fun you $\boldsymbol{c}$**.** Do to $\boldsymbol{a}$ as **1** made $\boldsymbol{b}$. I know $2 \times {}^{-}3 = {}^{-}6$ because the *Multiplier* $^{-}3$ leads us to subtract the *Multiplicand* 2 three times in succession from zero which gives us $0 - 2 - 2 - 2 = 0 - 6 = {}^{-}6$. Yet how will your triangles solve this?

$^{+}2 \times {}^{-}3 = c$



WALLY: The positive multiplier $^{+}3$ is above the origin so the negative multiplier $^{-}3$ is below the origin. When a number has no sign, we say it is positive. So again, we start at our fixed *Unit* 1 and draw a line to our *Multiplier*, which is located 3 unit lengths below the origin. Then, from 2 we…

ROOMY: Me! Let ME solve the puzzle! We draw a parallel line from 2 and it lands on the *Product* $^{-}6$. So, again, we see how, for $\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{c}$, as **1** is to $\boldsymbol{b}$, so $\boldsymbol{a}$ is to $\boldsymbol{c}$ [**Fig. R2**].

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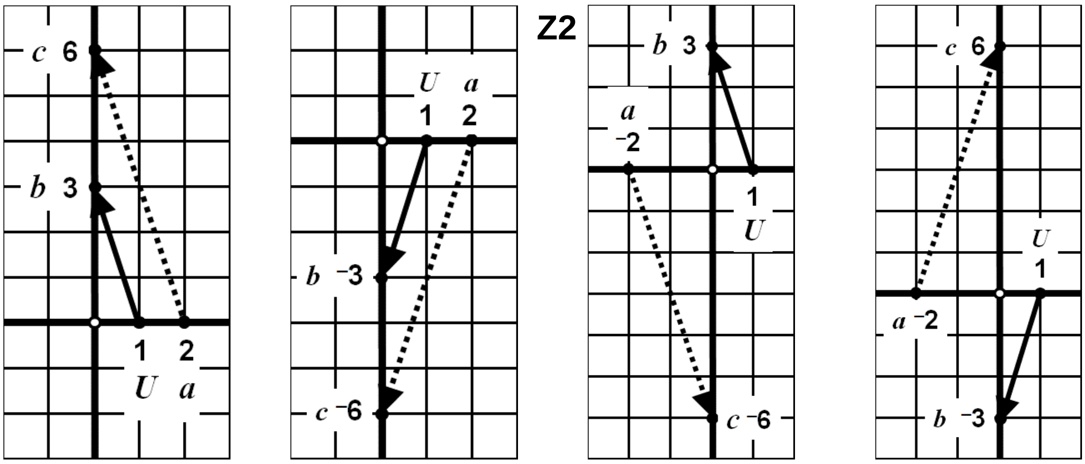
ZACKY: I will draw the four combinations of 2 × 3. Then, with our triangles on Podo's Paddock, we will see how like signs make $^{+}$positive and unlike signs make $^{-}$negative [**Fig. Z2**].

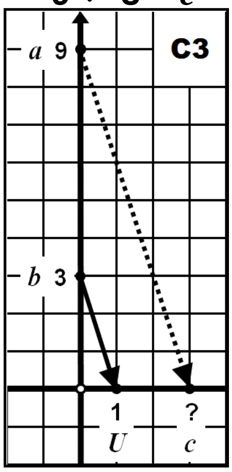
$^{+}2 \times {}^{+}3 = {}^{+}6$ | **$^{+}2 \times {}^{-}3 = {}^{-}6$** | **$^{-}2 \times {}^{+}3 = {}^{-}6$** | **$^{-}2 \times {}^{-}3 = {}^{+}6$**



As 1 is to $^{+}3$ so $^{+}2$ is to $^{+}6$ | As 1 is to $^{-}3$ so $^{+}2$ is to $^{-}6$ | As 1 is to $^{+}3$ so $^{-}2$ is to $^{-}6$ | As 1 is to $^{-}3$ so $^{-}2$ is to $^{+}6$

CARTY: With the *Dividend* $\boldsymbol{a}$ and the *Divisor* $\boldsymbol{b}$, it is easy to find the *Quotient*, $\boldsymbol{c}$. [Hint. Get *Dividends* and *Divisors* from a multiplication table to give students aliquot division.] We solve for $\boldsymbol{a} \div \boldsymbol{b} = \boldsymbol{c}$ with the example $^{+}9 \div {}^{+}3 = \boldsymbol{c}$. *'Division you* $\boldsymbol{c}$ *is lots of fun. Do to* $\boldsymbol{a}$ *as* $\boldsymbol{b}$ *made 1.'* We ask: as $^{+}3$ is to 1, so 9 is to ***what***? First, we draw a line from $\boldsymbol{b}$, the *Divisor*, which is $^{+}3$, to the *Unit* 1. Then we draw a parallel line from the *Dividend* $^{+}9$ to find the *Quotient* $\boldsymbol{c}$ [**Fig. C3**].

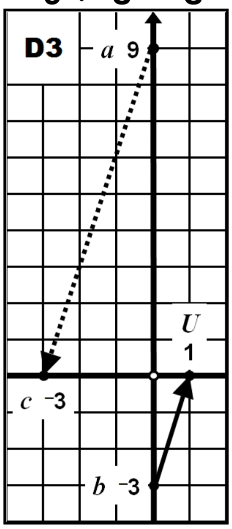
$^{+}9 \div {}^{+}3 = c$



ZACKY: Teacher can't explain division with positive *Dividends* and negative *Divisors*. In $^{+}9 \div {}^{-}3 = \boldsymbol{c}$ repeated subtraction fails, as you can't keep subtracting $^{-}3$ from $^{+}9$ to get to zero. Also, you can't have a negative number of equal groups!

The *Dividend* and *Divisor* vary proportionally.

$^{+}9 \div {}^{-}3 = {}^{-}3$



GERRY: The arrows convey or convect a number from point to point. In my Latin, vehĕre means to carry. That's why I call such arrows 'vectors'.

ROOMY: With $^{+}9 \div {}^{+}3 = \boldsymbol{c}$, I say 'a quantity [3], which divides a quantity [9], yields as its quotient [$\boldsymbol{c}$] always a number [i.e. 3], which is to the unit [1] as the divided quantity [9] is to the dividing quantity [3]', (Bockstaele, 2009).

DIO: In $^{+}9 \div {}^{-}3 = \boldsymbol{c}$ we ask, 'What did we do to $^{-}3$ to make 1?' We took one of three equal parts of $^{-}3$ which is $^{-}1$ and changed its sign to make 1. Having DONE THAT, we DO THIS! We take one of three equal parts of $^{+}9$ which is $^{+}3$ and change its sign to get $^{-}3$. The triangles agree [**Fig. D3**].

BRAMMY: Unseen for centuries, math is prettier and easier

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when numbers, magnitudes and things are balanced in mutual opposition around zero. To draw the multiplication 'meta-model' as **proportional covariation** (PCV), the leading solid arrow goes 1 to $\boldsymbol{b}$. The dashed arrow that follows goes $\boldsymbol{a}$ to $\boldsymbol{c}$. With the opposite operation of division, the leading arrow goes the opposite way! With division as **proportional covariation** (PCV), the solid leading arrow goes $\boldsymbol{b}$ to 1. The dashed arrow that follows goes $\boldsymbol{a}$ to $\boldsymbol{c}$. As Gerry's mother said, the triangles are similar with parallel vectors as hypotenuses.

CARTY: A negative *Dividend* $\boldsymbol{a}$ divided by a negative *Divisor* $\boldsymbol{b}$ results in a positive *Quotient* $\boldsymbol{c}$. As before, in $^{-}9 \div {}^{-}3 = \boldsymbol{c}$ we ask, 'What did we do to $^{-}3$ to make 1?' We took one of three equal parts of $^{-}3$ which is $^{-}1$ and changed its sign to make 1. Having DONE THAT, we DO THIS! We take one of three equal parts of $^{-}9$ which is $^{-}3$ and change its sign to $^{+}3$. [Can you draw this?]

BRAMMY: Teacher fails to explain fractional *Multipliers.* The denominator, or bottom number, gives us the number of equal parts the *Multiplicand* is partitioned or split into. The denominator is signless. The numerator, or top number (with sign), of a fractional *Multiplier*, tells us how many of the equal parts of the *Multiplicand* are to be added to zero or subtracted from zero.

HUIEY: Teacher says 2/3 is the *Multiplier* two-thirds, yet doesn't give us step by step instructions for what we are to do. I write $^{+}2/3$, as, of three equal parts of the *Multiplicand*, we add two equal parts to zero. So, in $^{+}6 \times {}^{+}2/3$, of three equal parts of six, we add two equal parts to zero. The denominator /3 creates three equal parts of positive six which are $^{+}2$, $^{+}2$ and $^{+}2$. The numerator $^{+}2/$ says we add two of those three equal parts to zero. So, we get $0 + {}^{+}2 + {}^{+}2$ or 4 as the answer to $^{+}6 \times {}^{+}2/3$.

BRAMMY: If we have $^{+}12 \times {}^{-}3/4$, the denominator /4 creates the four equal parts of twelve, $^{+}3$, $^{+}3$, $^{+}3$, and $^{+}3$. The numerator $^{-}3/$ says we must subtract three of those four equal parts from zero. So, we get $0 - {}^{+}3 - {}^{+}3 - {}^{+}3$, like bricks from zero, giving $0 - {}^{+}9$ which is $^{-}9$ and the answer to $^{+}12 \times {}^{-}3/4$.

ZACKY: 'Multiplication is also made use of in fractions and surds, to find a new quantity [product] in the same ratio (whatever it be) to the multiplicand, as the multiplier has to unity' (Newton, 1720). We can make triangles with 0 and any two points, one on each axis. [i.e. The PCV meta-model for multiplication and division applies to the *Reals*.]

ROOMY: I pay attention to what you say and get better at math the more I play! Now I know the figures of speech, it's not our fault, they failed to teach.

HUIEY: People fear math because False Explanations Appear Real [F.E.A.R.]. Yet math is a game to play and our mind is the playground!

ARCY: Eureka! Come play! Circles **can** be used to multiply, (Crabtree, 2016).

ZACKY: Teacher please read, we are the clients. We must sit high on shoulders of giants. Three cheers for Huiey Brammy China India and you!

ALL: **Hip Hip Hooray! Hip Hip Hooray! Hip Hip Hooray!**

[DONG DONG]

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Background and Discussion

In 1968, at age 7, I argued with my teacher over the definition of multiplication.

An explanation of $a \times b$ reads 'to multiply a by integral b is to add a to itself b times' (Borowski & Borwein, 2012). For 2 × 3, Grade 2C was asked: *'What is two added to itself three times?'* I correctly answered 8 yet was told I was wrong! Two added to 1 three times is written 1 + 2 + 2 + 2 = 7. So, two added to itself (2) three times is 2 + 2 + 2 + 2 = 8. Given the definition of multiplication above, my answer ought to have been right! To prove me wrong, my teacher drew three hops of 2 on the blackboard number line that landed on 6. Whilst right I was left confused. Eight was the answer yet my teacher was the expert.

My cognitive dissonance was resolved as my confidence dissolved. I must be stupid. A few years later, I feared and failed mathematics and repeated a year of school. Later, in 1983 age 21, I set a goal of explaining basic math in ways I might have liked age 7 to 12. Sensing yet not knowing what I was looking for, over the years, I began assembling the 10,000-piece jigsaw puzzle I imagined elementary math to be. I had no picture to follow, not even an outline. Many fail math, but could I fix it? To find out, I explored textbooks from Singapore, Japan and India and hundreds of old math books spanning 16 languages.

If only Brammy said, *'Three hops of two on number lines start at zero because 2 × 3 equals 0 + 2 + 2 + 2 or two added to zero (not itself) three times'.* If only Huiey had the 17th century *Suan Fa Yuan Ben* (Elements of Calculation), and read, *'…adding two to itself three times must be equal to eight'* (Jami, 2012).

I finally proved the multiplication definition I struggled with at age 7 (cited since 1570 as Euclid's) was created by a London haberdasher (Crabtree, 2016). Addition is binary needing two terms, so you can't add a number to itself. Sextus Empiricus (c. 200 CE) wrote 'What is added is different from that to which it is added and nothing is different from itself' (Bury, 1933). Multiplication is also binary. Perhaps I might have received better math lessons 200 years earlier. From the 1768 *Encyclopedia Britannica*, we read [with clarifications]:

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> *Multiplication by a positive Number implies a repeated Addition* [to zero] *But Multiplication by a Negative* [number] *implies a repeated Subtraction* [from zero]. *And when* $^{+}\boldsymbol{a}$ *is to be multiplied by* $^{-}\boldsymbol{n}$*, the Meaning is that* $^{+}\boldsymbol{a}$ *is to be subtracted* [from zero] *as often as there are Units in* $\boldsymbol{n}$*: Therefore the Product is negative, being* $^{-}\boldsymbol{na}$, (Bell & Macfarquhar, 1768).

Teachers are stuck in **Q1** because multiplication and addition connect via the distributive law. Yet multiplication also distributes over subtraction. We start at 0 and successively count different *Units* of 1 to define $\boldsymbol{a}$. (In life, the unit is $^{+}$ve or $^{-}$ve yet in math we default to $^{+}$ve.) From $\boldsymbol{a}$, we count on different *Units* of 1 $\boldsymbol{b}$ times to define $\boldsymbol{a} + \boldsymbol{b}$. We can also start at 0 and count different lots of $\boldsymbol{a}$, $\boldsymbol{b}$ times to define $\boldsymbol{a} \times \boldsymbol{b}$. From Brahmagupta's zero, $(0 = {}^{-}\boldsymbol{a} + {}^{+}\boldsymbol{a})$, just as $^{+}\boldsymbol{a}$ may be defined as $0 + \boldsymbol{a}$, so $^{-}\boldsymbol{a}$ may be defined as $0 - \boldsymbol{a}$. Because $\boldsymbol{a} \times {}^{+}3 = \boldsymbol{a} \times (0 + 3)$ and given $3 = 1 + 1 + 1$, this means $\boldsymbol{a} \times {}^{+}3 = \boldsymbol{a} \times (0 + 1 + 1 + 1)$. Conversely, we also discover $\boldsymbol{a} \times {}^{-}3 = \boldsymbol{a} \times (0 - 1 - 1 - 1)$. Therefore, **integer multiplication cannot be <u>defined</u> as 'repeated addition'.**

$$\boldsymbol{a} \times {}^{+}3 = \boldsymbol{a} \times (0 + 1 + 1 + 1) = 0 + \boldsymbol{a} + \boldsymbol{a} + \boldsymbol{a}$$

$$\boldsymbol{a} \times {}^{-}3 = \boldsymbol{a} \times (0 - 1 - 1 - 1) = 0 - \boldsymbol{a} - \boldsymbol{a} - \boldsymbol{a}.$$

$$\boldsymbol{a}^{+3} = 1 \times \boldsymbol{a} \times \boldsymbol{a} \times \boldsymbol{a}$$
$$\boldsymbol{a}^{+2} = 1 \times \boldsymbol{a} \times \boldsymbol{a}$$
$$\boldsymbol{a}^{+1} = 1 \times \boldsymbol{a}$$
$$\boldsymbol{a}^{0} = 1$$
$$\boldsymbol{a}^{-1} = 1 \div \boldsymbol{a}$$
$$\boldsymbol{a}^{-2} = 1 \div \boldsymbol{a} \div \boldsymbol{a}$$
$$\boldsymbol{a}^{-3} = 1 \div \boldsymbol{a} \div \boldsymbol{a} \div \boldsymbol{a}$$

Despite what mathematics dictionaries say, $\boldsymbol{a}^3$ or $\boldsymbol{a}$ cubed is <u>not</u> 'the result of multiplying a number, quantity or expression by itself three times' (Borowski & Borwein, 2012). Having returned the additive identity 0 to its premier place in integral multiplication, like Gauss, I returned the multiplicative identity 1 to its premier place within integral exponentiation (Gauss, 1801). Unlike Gauss, I extended the pattern to see the symmetry. Tablets evolved from slate to silicon. So why not evolve our elementary math foundations, by unifying ideas of ancient Greece, China and India? Proportional covariation (PCV), oppositional symmetry and zero unlock mathematics. All else is logic and algorithms (calculation). My goal with this play is to help every child have more math fun. I hope it made you smile and look forward to your questions, corrections and suggestions. Free resources also await you at www.jonathancrabtree.com/FIPP Thank you for reading.

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